ESTIMATION OF SCALE PARAMETER OF CAMBANIS-TYPE BIVARIATE UNIFORM AND EXPONENTIAL DISTRIBUTION USING CENSORED RANKED SET SAMPLES

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Abstract

The Cambanis family constitutes a crucial class of multivariate distributions, characterized by a broader range of correlation than the traditional families. This study explores the concept of Ranked Set Sampling (RSS), a technique applicable when ranking a set of sampling units can be easily achieved through a judgment method or an auxiliary variable. We examine a study variable, Y, correlated with the auxiliary variable, X, used for ranking the sampling units. We assume that the pair (X,Y) follows a Cambanis-type bivariate uniform (CTBU) and Cambanis-type bivariate exponential (CTBE) distribution. The primary focus of the paper is on the estimation of the scale parameter associated with the study variable in the CTBU and CTBE distributions using censored ranked set samples. Additionally, we improve estimation approach by deriving the best linear unbiased estimator (BLUE) of the scale parameter using censored RSS. A numerical efficiency comparison study is conducted and the results are presented.

Key words: Ranked set sampling, Concomitants of order statistics, Best linear unbiased estimator, Cambanis type bivariate uniform and exponential distribution

1. INTRODUCTION

Statistical distributions play a crucial role in the modeling and comprehension of intricate real-world phenomena. When dealing with bivariate data modeling, it becomes advantageous to explore families of bivariate distributions characterized by predefined marginal distributions, particularly when prior information is available in the form of these marginals. To address this, Morgenstern (1956) introduced a flexible family of distributions tailored for such scenarios. However, a notable constrains of the Morgenstern family is its confinement of the correlation coefficient to a relatively narrow range $\left(-\frac{1}{3},\frac{1}{3}\right)$. Consequently, this family is most suitable for modeling data with low correlation between variables. To overcome this limitation and broaden the spectrum of correlation between variables, several modifications to the Morgenstern family have been proposed in the existing literature. Noteworthy contributors including Sarmanov (1794), Cambanis (1977), Huang and Kotz (1984, 1999), Bairamov et al. (2001), Bairamov and Kotz (2002), and Veena and Thomas (2008), have extended the Morgenstern family. In particular, Cambanis (1977) introduced a family of distributions that serves as a natural generalization of the Morgenstern family. Nair et al. (2016) delved into an exploration of the distributional characteristics, dependence nature, reliability properties, and applications of the Cambanis family, showcasing its superiority in improving dependence coefficients compared to the Morgenstern family. Koshti and Kamalja (2021) derived an estimator for the scale parameter associated with the study variable for Cambanis-type bivariate uniform distribution (CTBU) based on different ranked set sampling (RSS) schemes. Alawady et. al. (2021) studied the concomitants of generalized order statistics and dual generalized order statistics within the Cambanis family of bivariate distributions, specifically considering nonzero parameter values. For additional advancements related to the Cambanis family of bivariate distributions, see, Koshti (2021), Kamalja and Koshti (2022), Husseiny et al. (2022), Chacko and George (2023) and George and Chacko (2023).

The RSS technique first introduced by McIntyre (1952) and is recommended when the variable of interest is difficult or expensive to measure but could be easily or cheaply ordered. RSS aims to enhance the precision of the sample mean, an estimator of population mean. Under specific conditions, the RSS scheme provides an effective way to achieve an observational economy. In implementing RSS, McIntyre employed the judgment method for ranking a set of sample units. However, the judgment method becomes impractical in situation where distinguishing the rank of one unit from another is ambiguous. Recognizing the limitations of imperfect ranking in RSS and its consequent impact on larger mean square errors of RSS estimators, Stokes (1977) proposed an alternative approach. In Stokes' framework, an auxiliary variable is introduced to rank sampling units. This approach is particularly useful in situations where the variable of interest, Y is difficult or costly to measure, but an auxiliary variable X, correlated with Y, is easily measurable and can be precisely ordered. The procedure of RSS using auxiliary variable described by Stokes (1977) is as follows.

Choose independent bivariate samples each of size n and observe the value of the auxiliary variable X on each of these units. For the first sample, select that unit for which the measurement on the auxiliary variable X is the smallest and measure the Y variate associated with it. In the second sample, choose Y associated to the second smallest X. This procedure is repeated until Y associated with the largest X in the last sample is measured. The resulting set of n units is called RSS. Let $(X_{(r)r}, Y_{[r]r}), r = 1,2, \dots, n$ be the pair selected from the r^{th} sample, where $X_{(r)r}$ denote the r^{th} order statistic of the auxiliary variable in the r^{th} sample and $Y_{[r]r}$ denote the measurement made on the Y variate associated with $X_{(r)r}$. David and Nagaraja (2003) referred $Y_{[r]r}$ as the concomitant of the r^{th} order statistic arising from the r^{th} sample. To deal with inference problems based on the RSS method proposed by Stokes (1977) requires the distribution theory of COS. Recently, Koshti and Kamalja (2023) highlighted the most important application of COS in RSS. Now, we briefly review some developments in estimation under censored RSS.

Chacko and Thomas (2008), Singh and Mehta (2015), Koshti and Kamalja (2017), Kamalja and Koshti (2019)and Irshad et al. (2023) obtained the BLUE of the scale parameter of the study variable in Morgenstern type bivariate (MTB) exponential, MTB uniform, Bivariate Lomax, MTB generalized uniform and MTB Bilal distribution respectively based on censored RSS.

In this paper, we estimate the scale parameter associated with study variable in the CTBU and Cambanis-type bivariate exponential (CTBE) distributions based on censored ranked set samples. Section 2, provides a comprehensive review of the CTBU and CTBE distribution in the context of COS. In section 3, we obtain the unbiased estimator and BLUE of the scale parameter associated with the study variable for CTBU and CTBE distribution based on censored RSS. We perform an efficiency comparison of estimators numerically in section 4. Finally, we conclude the paper in section 5.

2. CAMBANIS-TYPE BIVARIATE DISTRIBUTIONS

In this section, we briefly discuss the family of Cambanis type bivariate distribution introduced by Cambanis (1977). Additionally, we present brief summary on COS for both CTBU and CTBE distribution.

The distribution function of the Cambanis-type bivariate distribution for a bivariate random variable (X, Y) with parameters $\alpha_1, \alpha_2, \alpha_3$ $\left(CTB(\alpha_1, \alpha_2, \alpha_3\,)\right)$, as provided by Cambanis (1977) is,

$$H_{X,Y}\,(x, y) = F_X(x)F_Y(y)[1 + \alpha_1\,\{1 - F_X\,(x)\,\} + \alpha_2\,\{1 - F_Y\,(y)\,\} + \alpha_3\,\{1 - F_X\,(x)\,\}\{1 - F_Y\,(y)\,\}],$$

where the parameters α_1, α_2 and α_3 are real constants that must satisfy the following conditions.

$$1 + \alpha_1 + \alpha_2 + \alpha_3 \geq 0, 1 + \alpha_1 - \alpha_2 - \alpha_3 \geq 0,$$

$$1 - \alpha_1 + \alpha_2 - \alpha_3 \geq 0, 1 - \alpha_1 - \alpha_2 + \alpha_3 \geq 0.$$

The Cambanis family for bivariate distributions reduces to the Morgenstern family when both α_1 and α_2 are zero. Scaria and Nair (1999) examined the distribution theory on COS from Morgenstern family, while Thomas (2018) focused on the Cambanis family. Thomas (2018) derived the COS for the general $CTB(\alpha_1, \alpha_2, \alpha_3)$distribution and for the $CTBE(\alpha_1, \alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution, specifically when $\alpha_1 = 0$.

2.1 CAMBANIS-TYPE BIVARIATE UNIFORM DISTRIBUTION

We consider a CTBU distribution with parameters $(\alpha_2, \alpha_3, \theta_1, \theta_2)$, denoted as $CTBU(\alpha_2, \alpha_3, \theta_1, \theta_2)$. Let $(X_i, Y_i), i = 1, 2, \dots, n$ be a random sample of size n from $CTBU(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution with probability density function (pdf) is given by,

$$h(x, y) = \frac{1}{\theta_1\theta_2}\left[1 + \alpha_2\left(1 - \frac{2y}{\theta_2}\right) + \alpha_3\left(1 - \frac{2x}{\theta_1}\right)\left(1 - \frac{2y}{\theta_2}\right)\right]; 0 < x < \theta_1, 0 < y < \theta_2,$$

where $|\alpha_2 + \alpha_3| \leq 1, |\alpha_2 - \alpha_3| \leq 1, \theta_1, \theta_2 > 0$.

Thomas and Scaria (2011) studied and obtained the COS for $CTB(0, \alpha_2, \alpha_3)$ distribution and Koshti and Kamalja (2021) obtained the pdf $h_{Y_{[r]n}}(y)$ of concomitants of r^{th} order statistic for CTBU distribution as,

$$h_{Y_{[r]n}}(y) = \frac{1}{\theta_2}\left[1 + \left(\alpha_2 + \alpha_3\frac{n-2r+1}{n+1}\right)\left(1 - \frac{2y}{\theta_2}\right)\right]\;; 0 < y < \theta_2.$$

The mean and variance of $Y_{[r]n}$ are,

$$E\left(Y_{[r]n}\right) = \theta_2\xi_r, \qquad Var\left(Y_{[r]n}\right) = \theta_2^2\delta_r\,, \tag{1}$$

where $\xi_r = \frac{1}{2}\left[1 - \frac{1}{3}\left(\alpha_2 + \alpha_3\frac{n-2r+1}{n+1}\right)\right]$, $\qquad \delta_r = \frac{1}{12}\left[1 - \frac{1}{3}\left(\alpha_2 + \alpha_3\frac{n-2r+1}{n+1}\right)^2\right]$.

The above expression of mean and variance of concomitants of r^{th} order statistic $Y_{[r]n}$ would be used for estimating scale parameter θ_2 under censored ranked set samples.

2.2 CAMBANIS-TYPE BIVARIATE EXPONENTIAL DISTRIBUTION

Let $(X_i, Y_i), i = 1, 2, \ldots, n$ be a random sample of size n from Cambanis-type bivariate uniform distribution with parameters $(\alpha_2, \alpha_3, \theta_1, \theta_2)$ denoted as $CTBE(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution with pdf given by,

$$h(x, y) = \frac{e^{-\left(\frac{x}{\theta_1}+\frac{y}{\theta_2}\right)}}{\theta_1\theta_2}\left[1 + \alpha_2\left(2e^{\frac{-y}{\theta_2}} - 1\right) + \alpha_3\left(2e^{\frac{-x}{\theta_1}} - 1\right)\left(2e^{\frac{-y}{\theta_2}} - 1\right)\right];$$

$$x, y > 0, \theta_1, \theta_2 > 0, |\alpha_2 + \alpha_3| \leq 1, |\alpha_2 - \alpha_3| \leq 1.$$

The pdf $h_{Y_{[r]n}}(y)$ of concomitant of r^{th} order statistic from a sample of size $n,$ drawn from the $CTBE(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution, where$1 \leq r \leq n$, is given by,

$$h_{Y_{[r]n}}(y) = \frac{e^{\frac{-y}{\theta_2}}}{\theta_2}\left[1 + \left\{\alpha_2 + \alpha_3\frac{n-2r+1}{n+1}\right\}\left\{2e^{\frac{-y}{\theta_2}} - 1\right\}\right]; \quad y > 0, \theta_2 > 0.$$

The mean and variance of $Y_{[r]n}$ are,

$$E(Y_{[r]n}) = \theta_2\xi_r, \qquad Var(Y_{[r]n}) = \theta_2^2\delta_r, \tag{2}$$

where $\xi_r = 1 - \frac{1}{2}\left(\alpha_2 + \alpha_3\frac{n-2r+1}{n+1}\right), \delta_r = 1 - \frac{1}{2}\left(\alpha_2 + \alpha_3\frac{n-2r+1}{n+1}\right) - \frac{1}{4}\left(\alpha_2 + \alpha_3\frac{n-2r+1}{n+1}\right)^2$.

3. ESTIMATOR OF SCALE PARAMETER BASED ON CENSORED RSS

In this section we discuss in brief the steps required to obtain the unbiased estimator of scale parameter θ_2 and BLUE of θ_2 based on censored ranked set samples.

In an example discussed by Bain (1978) concerning the pollution of sea water due to oil, the tar deposits near the seashore are used as an auxiliary variable. When there is no tar deposit at the seashore, the corresponding sea water samples cannot be measured and they are treated as censored. For ranking X

observations in a sample, the censored units are assumed to have distinct and consecutive lower ranks, while the remaining units are ranked with the next higher ranks in a natural order. In the censored scheme of RSS, where k units are censored, we denote the RSS observations in the sample as $\rho_1 Y_{[1]1}, \rho_2 Y_{[2]2}, \dots, \rho_n Y_{[n]n}$ where,

$$\rho_i = \begin{cases} 0 & \text{if the } i^{th} \text{ unit is censored} \\ 1 & \text{otherwise} \end{cases}$$

such that $\sum_{i=1}^{n} \rho_i = n - k$.

In this case the censored RSS mean, denoted as $\bar{y}_{RSS}^{c}$, is defined as follows,

$$\bar{y}_{RSS}^{c} = \frac{\sum_{i=1}^{n} \rho_i Y_{[i]i}}{n-k}.$$

It may be noted that $\rho_i = 0$ need not occur in a natural order for $i = 1, 2, \dots, n$. Let the integers $m_1, m_2, \dots, m_{n-k}$ be such that $1 \le m_1 < m_2 < \cdots < m_{n-k} \le n$ and $\rho_{m_i} = 1$. Then,

$$E[\bar{y}_{RSS}^{c}] = \frac{\theta_2}{n-k} \sum_{i=1}^{n-k} \xi_{m_i} \,,$$

where $$\xi_{m_i} = \begin{cases} \frac{1}{2}\left[1 - \frac{1}{3}\left(\alpha_2 + \alpha_3 \frac{n-2m_i+1}{n+1}\right)\right] & \text{; for CTBU distribution (using (1))} \\ 1 - \frac{1}{2}\left(\alpha_2 + \alpha_3 \frac{n-2m_i+1}{n+1}\right) & \text{; for CTBE distribution (using (2))} \end{cases}.$$

Even though the RSS mean in the censored case is not an unbiased estimator for θ_2, we can construct an unbiased estimator based on $\bar{y}_{RSS}^{c}$. Let $Y_{[m_i]m_i}, i = 1,2, \dots, n-k$, be the censored RSS observations. When α_2 and α_3 are known, an unbiased estimator $\hat{\theta}_{2,RSS}^{c}(k)$ of θ_2 based on censored RSS of size k and its variance is given by,

$$\hat{\theta}_{2,RSS}^{c}(k) = \frac{(n-k)\bar{y}_{RSS}^{c}}{\sum_{i=1}^{n-k} \xi_{m_i}} \quad \text{and} \quad Var\left(\hat{\theta}_{2,RSS}^{c}(k)\right) = \theta_2^2 \frac{\sum_{i=1}^{n-k} \delta_{m_i}}{\left(\sum_{i=1}^{n-k} \xi_{m_i}\right)^2},$$

where,

$$\delta_{m_i}=\begin{cases}\dfrac{1}{12}\left[1-\dfrac{1}{3}\left(\alpha_2+\alpha_3\dfrac{n-2m_i+1}{n+1}\right)^2\right] & \text{; for CTBU distribution (using (1))}\\ 1-\dfrac{1}{2}\left(\alpha_2+\alpha_3\dfrac{n-2m_i+1}{n+1}\right)-\dfrac{1}{4}\left(\alpha_2+\alpha_3\dfrac{n-2m_i+1}{n+1}\right)^2 & \text{; for CTBE distribution (using (2))}\end{cases}$$

We now propose the BLUE of θ_2 based on the censored RSS. Let $\boldsymbol{Y}_{[\boldsymbol{n}]}(k)=\left(Y_{[m_1]m_1}\quad Y_{[m_2]m_2}\quad\cdots\quad Y_{[m_{n-k}]m_{n-k}}\right)'$ be the vector of observations in censored RSS. The mean vector and the dispersion matrix of $\boldsymbol{Y}_{[\boldsymbol{n}]}(k)$ are given by,

$$E\left(\boldsymbol{Y}_{[\boldsymbol{n}]}(k)\right)=\theta_2\boldsymbol{\xi}(\boldsymbol{k}), \tag{3}$$

$$D\left(\boldsymbol{Y}_{[\boldsymbol{n}]}(k)\right)=\theta_2^2G(k), \tag{4}$$

where $\boldsymbol{\xi}(\boldsymbol{k})=\left(\xi_{m_1}\quad \xi_{m_2}\quad\cdots\quad \xi_{m_{n-k}}\right)'$ and $G(k)=diag\left(\delta_{m_1},\delta_{m_2},\ldots,\delta_{m_{n-k}}\right)$.

If $\alpha_2$and α_3 involved in $\boldsymbol{\xi}(\boldsymbol{k})$and $G(k)$ are known, then (3) and (4) defines a generalized Gauss-Markov set up and hence the BLUE $\tilde{\theta}^c_{2,RSS}(k)$ of $\theta_2$is obtained as,

$$\tilde{\theta}^c_{2,RSS}(k)=\left(\boldsymbol{\xi}(\boldsymbol{k})'G(k)^{-1}\boldsymbol{\xi}(\boldsymbol{k})\right)^{-1}\boldsymbol{\xi}(\boldsymbol{k})'G(k)^{-1}\boldsymbol{Y}_{[\boldsymbol{n}]}(k), \tag{5}$$

and $$Var\left(\tilde{\theta}^c_{2,RSS}(k)\right)=\left(\boldsymbol{\xi}(\boldsymbol{k})'G(k)^{-1}\boldsymbol{\xi}(\boldsymbol{k})\right)^{-1}\theta_2^2. \tag{6}$$

Simplifying the expression from (5) and (6), we get,

$$\tilde{\theta}^c_{2,RSS}(k)=\sum_{i=1}^{n-k}a_{m_i}Y_{[m_i]m_i}, \qquad Var\left(\tilde{\theta}^c_{2,RSS}(k)\right)=\frac{\theta_2^2}{\sum_{i=1}^{n-k}(\xi^2_{m_i}/\delta_{m_i})},$$

where$a_{m_i}=\dfrac{\xi_{m_i}/\delta_{m_i}}{\sum_{i=1}^{n-k}\xi^2_{m_i}/\delta_{m_i}}$.

Remark 1. If we set $k=0$ and $m_i=r$, then the results presented in this paper simplifies to the complete (uncensored) ranked set sample scenario. Previously, this uncensored RSS was discussed for the CTBU distribution by Koshti and Kamalja (2021) and for the CTBE distribution by Kamalja and Koshti (2022).

Remark 2. In real-life situations, when applying the estimator of θ_2 for the CTBU or CTBE distribution, it is often necessary to estimate the parameters α_2 and $\alpha_3$first. The estimation procedure for α_2 and α_3 in the CTBU distribution has been detailed discussed by Koshti (2021) and for the CTBE distribution, refer to Kamalja and Koshti (2022).

4. RELATIVE EFFICIENCY

The relative efficiency(RE) of $\tilde{\theta}^{c}_{2,RSS}(k)$ (BLUE) over $\hat{\theta}^{c}_{2,RSS}(k)$ (censored RSS mean) is given by

$$RE = \frac{Var\left(\hat{\theta}^{c}_{2,RSS}(k)\right)}{Var\left(\tilde{\theta}^{c}_{2,RSS}(k)\right)} = \frac{\left(\sum_{i=1}^{n-k}\delta_{m_i}\right)\left(\sum_{i=1}^{n-k}\frac{\xi^2_{m_i}}{\delta_{m_i}}\right)}{\left(\sum_{i=1}^{n-k}\xi_{m_i}\right)^2}$$

We have computed the values of RE for different values of n, α_2 and α_3 for CTBU and CTBE distribution and presented in Table 1 and Table 2 respectively.Different trends can be seen in these numerically computed efficiencies. We can draw the following conclusions from Table 1 and Table 2.

- Observe that for some specific values of $n, k, \alpha_2$and α_3, $\tilde{\theta}^{c}_{2,RSS}(k)$ (BLUE) is efficient than $\hat{\theta}^{c}_{2,RSS}(k)$ (censored RSS mean) whereas for other values, both estimators are equally efficient.
- For $k = 0, RE|_{\alpha_3} = RE|_{-\alpha_3}$for fixed α_2 and $RE|_{\alpha_2} = RE|_{-\alpha_2}$ when $\alpha_3 = 0$.
- For $\alpha_2 = 0$, the obtained REunder CTBU (CTBE) distribution reduces to the efficiency expression under MTBU (MTBE) distribution, as studied by Singh and Mehta (2015) (Chacko and Thomas (2008)).

5. CONCLUSION

In this paper, we consider the Ranked set sampling (RSS) which is an efficient technique for estimating parameters and is applicable whenever ranking on a set of sampling units can be done easily by a judgment method or based on an auxiliary variable. Here, we assume (X, Y) follows the CTBU and CTBE distribution where a study variable Y is difficult and/or expensive to measure and is correlated with an auxiliary variable X that is readily measurable. This paper addresses the problem of estimation of scale parameter associated with the study variable Y based on censored ranked set samples for both the CTBU and CTBE distribution. Further, the BLUE of the scale parameter based on censored RSS is also obtained. The efficiency performance of proposed estimators is also studied numerically.

Table 1 The values of RE for $CTBU(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution

n, α_2	k	α_3	RE	n, α_3	k	α_2	RE	n, α_2	k	α_3	RE	n, α_3	k	α_2	RE
n: 10 α_2: 0	0	-1.0	1.0413	n: 20 α_3: 0	0	-1.0	1.0479	n: 10 α_2: 0.2	0	-0.8	1.0110	n: 12 α_3: 0.2	0	-0.8	1.0103
		-0.75	1.0201			-0.75	1.0228			-0.6	1.0055			-0.6	1.0062
		-0.50	1.0081			-0.50	1.0091			-0.4	1.0023			-0.4	1.0038
		-0.25	1.0019			-0.25	1.0021			-0.2	1.0005			-0.2	1.0023
		0.25	1.0019			0.25	1.0021			0.2	1.0005			0.2	1.0006
		0.50	1.0081			0.50	1.0091			0.4	1.0023			0.4	1.0001
		0.75	1.0201			0.75	1.0228			0.6	1.0055			0.6	1.0000
		1.0	1.0413			1.0	1.0479			0.8	1.0110			0.8	1.0006
	2	-1.0	1.0405		3	-1.0	1.0484		2	-0.8	1.0115		3	-0.8	1.0050
		-0.75	1.0183			-0.75	1.0218			-0.6	1.0054			-0.6	1.0031
		-0.50	1.0067			-0.50	1.0080			-0.4	1.0020			-0.4	1.0019
		-0.25	1.0014			-0.25	1.0017			-0.2	1.0004			-0.2	1.0011
		0.25	1.0010			0.25	1.0014			0.2	1.0003			0.2	1.0002
		0.50	1.0036			0.50	1.0050			0.4	1.0009			0.4	1.0000
		0.75	1.0073			0.75	1.0106			0.6	1.0018			0.6	1.0000
		1.0	1.0117			1.0	1.0186			0.8	1.0029			0.8	1.0005
	4	-1.0	1.0331		7	-1.0	1.0424		4	-0.8	1.0097		5	-0.8	1.0028
		-0.75	1.0140			-0.75	1.0178			-0.6	1.0042			-0.6	1.0017
		-0.50	1.0047			-0.50	1.0060			-0.4	1.0014			-0.4	1.0010
		-0.25	1.0009			-0.25	1.0012			-0.2	1.0003			-0.2	1.0006
		0.25	1.0005			0.25	1.0007			0.2	1.0001			0.2	1.0001
		0.50	1.0013			0.50	1.0019			0.4	1.0003			0.4	1.0000
		0.75	1.0019			0.75	1.0030			0.6	1.0004			0.6	1.0000
		1.0	1.0021			1.0	1.0038			0.8	1.0004			0.8	1.0004
	6	-1.0	1.0206		11	-1.0	1.0297		6	-0.8	1.0060		7	-0.8	1.0013
		-0.75	1.0081			-0.75	1.0116			-0.6	1.0025			-0.6	1.0008
		-0.50	1.0025			-0.50	1.0036			-0.4	1.0008			-0.4	1.0005
		-0.25	1.0004			-0.25	1.0006			-0.2	1.0001			-0.2	1.0003
		0.25	1.0002			0.25	1.0003			0.2	1.0000			0.2	1.0000
		0.50	1.0003			0.50	1.0005			0.4	1.0001			0.4	1.0000
		0.75	1.0002			0.75	1.0005			0.6	1.0000			0.6	1.0000
		1.0	1.0002			1.0	1.0004			0.8	1.0002			0.8	1.0002
	8	-1.0	1.0061		15	-1.0	1.0134		8	-0.8	1.0017		9	-0.8	1.0004
		-0.75	1.0022			-0.75	1.0048			-0.6	1.0007			-0.6	1.0002
		-0.50	1.0006			-0.50	1.0014			-0.4	1.0002			-0.4	1.0001
		-0.25	1.0001			-0.25	1.0002			-0.2	1.0000			-0.2	1.0001
		0.25	1.0000			0.25	1.0001			0.2	1.0000			0.2	1.0000
		0.50	1.0000			0.50	1.0001			0.4	1.0000			0.4	1.0000
		0.75	1.0000			0.75	1.0000			0.6	1.0000			0.6	1.0000
		1.0	1.0002			1.0	1.0004			0.8	1.0002			0.8	1.0001

Table 2 The values of RE for $CTBE(\alpha_2, \alpha_3, \theta_1, \theta_2)$ distribution

n, α_2	k	α_3	RE	n, α_3	k	α_2	RE	n, α_2	k	α_3	RE	n, α_3	k	α_2	RE
$n: 10$ $\alpha_2: 0$	0	-1.0	1.0067	$n: 20$ $\alpha_3: 0$	0	-1.0	1	$n: 10$ $\alpha_2: 0.2$	0	-0.8	1.0102	$n: 12$ $\alpha_3: 0.2$	0	-0.8	1.0009
		-0.75	1.0016			-0.75	1			-0.6	1.0033			-0.6	1.0005
		-0.50	1.0003			-0.50	1			-0.4	1.0009			-0.4	1.0003
		-0.25	1.0000			-0.25	1			-0.2	1.0002			-0.2	1.0001
		0.25	1.0000			0.25	1			0.2	1.0002			0.2	1.0002
		0.50	1.0003			0.50	1			0.4	1.0009			0.4	1.0011
		0.75	1.0016			0.75	1			0.6	1.0033			0.6	1.0044
		1.0	1.0067			1.0	1			0.8	1.0102			0.8	1.0190
	2	-1.0	1.0019		3	-1.0	1		2	-0.8	1.0011		3	-0.8	1.0004
		-0.75	1.0006			-0.75	1			-0.6	1.0005			-0.6	1.0003
		-0.50	1.0001			-0.50	1			-0.4	1.0002			-0.4	1.0001
		-0.25	1.0000			-0.25	1			-0.2	1.0001			-0.2	1.0000
		0.25	1.0000			0.25	1			0.2	1.0002			0.2	1.0002
		0.50	1.0003			0.50	1			0.4	1.0010			0.4	1.0008
		0.75	1.0018			0.75	1			0.6	1.0040			0.6	1.0032
		1.0	1.0085			1.0	1			0.8	1.0132			0.8	1.0146
	4	-1.0	1.0022		7	-1.0	1		4	-0.8	1.0002		5	-0.8	1.0002
		-0.75	1.0008			-0.75	1			-0.6	1.0001			-0.6	1.0001
		-0.50	1.0002			-0.50	1			-0.4	1.0000			-0.4	1.0001
		-0.25	1.0000			-0.25	1			-0.2	1.0000			-0.2	1.0000
		0.25	1.0000			0.25	1			0.2	1.0001			0.2	1.0001
		0.50	1.0004			0.50	1			0.4	1.0010			0.4	1.0006
		0.75	1.0024			0.75	1			0.6	1.0043			0.6	1.0024
		1.0	1.0118			1.0	1			0.8	1.0155			0.8	1.0110
	6	-1.0	1.0017		11	-1.0	1		6	-0.8	1.0003		7	-0.8	1.0001
		-0.75	1.0006			-0.75	1			-0.6	1.0001			-0.6	1.0001
		-0.50	1.0001			-0.50	1			-0.4	1.0000			-0.4	1.0000
		-0.25	1.0000			-0.25	1			-0.2	1.0000			-0.2	1.0000
		0.25	1.0000			0.25	1			0.2	1.0001			0.2	1.0001
		0.50	1.0003			0.50	1			0.4	1.0007			0.4	1.0004
		0.75	1.0025			0.75	1			0.6	1.0035			0.6	1.0014
		1.0	1.0131			1.0	1			0.8	1.0146			0.8	1.0070
	8	-1.0	1.0005		15	-1.0	1		8	-0.8	1.0001		9	-0.8	1.0000
		-0.75	1.0002			-0.75	1			-0.6	1.0000			-0.6	1.0000
		-0.50	1.0000			-0.50	1			-0.4	1.0000			-0.4	1.0000
		-0.25	1.0000			-0.25	1			-0.2	1.0000			-0.2	1.0000
		0.25	1.0000			0.25	1			0.2	1.0000			0.2	1.0000
		0.50	1.0001			0.50	1			0.4	1.0002			0.4	1.0001
		0.75	1.0012			0.75	1			0.6	1.0014			0.6	1.0006
		1.0	1.0074			1.0	1			0.8	1.0070			0.8	1.0030

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