TWO PARAMETERS EXTENSION OF FUZZY SET: A REVIEW

Abhilash Kangsha Banik[1*], Palash Dutta[2]
Department of Mathematics, Dibrugarh University, PIN-786004, India
email: abhibanik99@gmail.com

Abstract

Uncertainty is inevitably present in every real-world problem. Various models have been developed to solve such problems. The Fuzzy Set Theory (FST) by L.A. Zadeh is the most notable among them. The concept of Intuitionistic Fuzzy Set (IFS) was developed as an improvement over FST. Due to its limitations, the concept of Pythagorean Fuzzy Set (PFS) was developed and the Fermatean Fuzzy Set (FFS). Also, the q-Rung Orthopair Fuzzy Set (q-ROFS) concept was developed as a generalization of IFS, PFS and FFS. A further generalized concepts of n, m-Rung Orthopair Fuzzy Set (n, m-ROFS) and Generalized Fuzzy Set (GFS) were developed as a generalization of q-Rung Orthopair Fuzzy Set (q-ROFS). Finally, the notion of Hyperbolic Fuzzy Set (HFS) was developed which serves as a better alternative to all the existing variants of Fuzzy Set. The aim here is to discuss the advantages of HFS citing the limitations of the existing approaches.

Key words: Fuzzy Set. q-ROFS. GFS. HFS.

1. INTRODUCTION

Decision-making is a significant feature of most problems due to the presence of uncertainty in them. Zadeh introduced Fuzzy Set Theory (FST) [7] to reduce uncertainty by introducing the concept of membership degree. The Fuzzy Set theory has been widely useful in pattern recognition, medical diagnosis, and decision-making problems.

However, it is not sufficient in some situations, so Attanosov introduced Intuitionistic Fuzzy Set (IFS) [1] by introducing non-membership degrees such that their sum is restricted by 1. However, IFS fails when the sum of the membership degree and the non-membership degree exceeds 1 and then Yager introduced the notion of Pythagorean Fuzzy Set (PFS) [7] such that the sum of the squares of the membership degree and the non-membership degree is restricted by 1. Similarly, Senapati and Yager introduced the notion of Fermatean Fuzzy Set (FFS) [6] such that the sum of the cubes of the membership degree and the non-membership degree is restricted by 1.

Further, Yager developed the concept of q-Rung Orthopair Fuzzy Set (q-ROFS) [8] as a generalization of IFS, PFS and FFS with the condition that the sum of q^{th} powers of the membership degree and the non-membership degree is restricted by 1 such that for q=1, q-ROFS becomes IFS, whereas for q=2, it becomes PFS and FFS for q=3. Hence, raising the powers of q increases the range of consideration considering the membership and non-membership degrees.

A further generalized concepts of n, m-Rung Orthopair Fuzzy Set (n, m-ROFS) [4] was developed by Ibrahim and Alshammari as a generalization of q-ROFS with the condition that the sum of n^{th} power of the membership degree and the m^{th}

power non-membership degree is restricted by 1 such that for n=m, n, m-ROFS becomes q-ROFS. Similarly, Generalized Fuzzy Set (GFS) [5] was developed by Munir et al. such that the sum of n^{th} power of the membership degree and the m^{th} power non-membership degree is restricted by 1. The basic difference between n, m-ROFS and GFS is in the hesitancy degree (degree of indeterminacy).

Two Factor Theory [3] proposed by Frederick Herzberg considered two factors to explain people's behavior at work, i.e., Motivating and Hygiene Factors. Motivating Factors leads to satisfaction, whereas not maintaining Hygiene Factors leads to dissatisfaction. Also, these two factors do not complement each other. Taking motivation from the Two Factor theory, when the membership degree (or the non-membership degree) is 1 and the non-membership (or the membership degree) is not zero, then q-ROFS, GFS and n, m-ROFS fails. Therefore, there is a need for a new approach that can solve the limitations of the existing approaches where more independence of selection of the membership degree and the non-membership degree can be considered.

The concept of Hyperbolic Fuzzy Set (HFS) [2] was developed as a solution, taking inspiration from the Two Factor Theory. It allows for a higher degree of independence in selecting degree of optimism and degree of pessimism. Unlike other frameworks such as q-ROFS, GFS, and n, m-ROFS, the concept of HFS enables the consideration of an degree of optimism of 1 while also incorporating a non-zero degree of pessimism (and vice versa). This concept of HFS has opened up numerous avenues for addressing decision-making problems. It provides decision-makers with enhanced opportunities to navigate complex scenarios and achieve optimal outcomes

In this study, we first give some relevant preliminary definitions. In the next section, the gradual development from IFS to q-ROFS, GFS and n, m-ROFS is discussed using geometric interpretations. The limitations of the existing approaches are shown with examples. Also, the necessity of HFS is given. Finally, a brief conclusion is presented in it.

2. PRELIMINARIES

Now we discuss about the preliminary definitions of various notions.

Definition (Fuzzy Set): [9] Let us consider X as a non-empty universal set. Then, we can define a Fuzzy Set in X by the expression $\mathcal{F} = \{<x, \mu_{\mathcal{F}}(x)> x \in X\}$; where the functions $\mu_{\mathcal{F}}(x): X \rightarrow [0,1]$ denotes the degree of membership of the element $x \in X$ to the set $\mathcal{F}$.

Definition (Intuitionistic Fuzzy Set): [1] Let us consider X as a non-empty universal set. Then, we can define an Intuitionistic Fuzzy Set in X by the expression $I=\{<x, \mu_I(x), \nu_I(x)> x \in X\}$; where the functions $\mu_I(x): X \rightarrow [0,1]$ denotes the degree of membership (support for membership) and $\nu_I(x): X \rightarrow [0,1]$ denotes degree of non-membership (support against membership) of the element $x \in X$ to the set I such that it satisfies the condition $0 \leq (\mu_I(x)) + (\nu_I(x)) \leq 1$.

The degree of indeterminacy of $x \in X$ to I is defined by $\pi_I(x) = [1- (\mu_I(x)+ \nu_I(x))]$ and $\pi_I(x) \in [0, 1]$ satisfying the condition $(\mu_I(x)) + (\nu_I(x)) + (\pi_I(x)) = 1$.

Definition (Pythagorean Fuzzy Set): [7] Let us consider X as a non-empty universal set. Then, we can define a Pythagorean Fuzzy Set in X by the expression $P=\{<x, \mu_P(x), \nu_P(x)> x \in X\}$; where the functions $\mu_P(x): X \rightarrow [0,1]$ denotes the degree of membership (support for membership) and $\nu_P(x)$:X→[0,1] denotes degree of non-membership (support against membership) of the element x ϵ X to the set P such that it satisfies the condition $0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1$.

The degree of indeterminacy of x ϵ X to P is defined by $\pi_P(x) = [1-\{(\mu_P(x))^2 + (\nu_P(x))^2\}]^{(1/2)}$ and $\pi_P(x)$ ϵ [0, 1] satisfying the condition $(\mu_P(x))^2 + (\nu_P(x))^2 + (\pi_P(x))^2 = 1$.

Definition (Fermatean Fuzzy Set): [6] Let us consider X as a non-empty universal set. Then, we can define a Fermatean Fuzzy Set in X by the expression F={<x, $\mu_F(x)$, $\nu_F(x)$>x ϵ X}; where the functions $\mu_F(x)$:X→ [0,1] denotes the degree of membership (support for membership) and $\nu_F(x)$:X→[0,1] denotes degree of non-membership (support against membership) of the element x ϵ X to the set F such that it satisfies the condition $0 \leq (\mu_F(x))^3 + (\nu_F(x))^3 \leq 1$.

The degree of indeterminacy of x ϵ X to F is defined by $\pi_F(x) = [1-\{(\mu_F(x))^3 + (\nu_F(x))^3\}]^{(1/3)}$ and $\pi_F(x)$ ϵ [0, 1] satisfying the condition $(\mu_F(x))^3 + (\nu_F(x))^3 + (\pi_F(x))^3 = 1$.

Definition (q-Rung Orthopair Fuzzy Set): [8] Let us consider X as a non-empty universal set. Then, we can define a q-Rung Orthopair Fuzzy Set in X by the expression Q={<x, $\mu_Q(x)$, $\nu_Q(x)$>x ϵ X}; where the functions $\mu_Q(x)$:X→ [0,1] denotes the degree of membership (support for membership) and $\nu_Q(x)$:X→[0,1] denotes degree of non-membership (support against membership) of the element x ϵ X to the set Q such that it satisfies the condition $0 \leq (\mu_Q(x))^q + (\nu_Q(x))^q \leq 1$.

The degree of indeterminacy of x ϵ X to Q is defined by $\pi_Q(x) = [1-\{(\mu_Q(x))^q + (\nu_Q(x))^q\}]^{(1/q)}$ and $\pi_Q(x)$ ϵ [0, 1] satisfying the condition $(\mu_Q(x))^q + (\nu_Q(x))^q + (\pi_Q(x))^q = 1$.

Definition (n, m-Rung Orthopair Fuzzy Set): [4] Let us consider X as a non-empty universal set. Then, we can define a n, m-Rung Orthopair Fuzzy Set in X by the expression N={<x, $\mu_N(x)$, $\nu_N(x)$>x ϵ X}; where the functions $\mu_N(x):X\rightarrow [0,1]$ denotes the degree of membership (support for membership) and $\nu_N(x):X\rightarrow[0,1]$ denotes degree of non-membership (support against membership) of the element x ϵ X to the set Q such that it satisfies the condition $0 \leq (\mu_N(x))^n + (\nu_N(x))^m \leq 1$.

The degree of indeterminacy of x ϵ X to N is defined by $\pi_N(x) = [1-\{(\mu_N(x))^n + (\nu_N(x))^m\}]^{(1/n+m)}$ and $\pi_N(x) \in [0, 1]$ satisfying the condition $(\mu_N(x))^n + (\nu_N(x))^m + (\pi_N(x))^{n+m} = 1$.

Definition (Generalized Fuzzy Set): [5] Let us consider X as a non-empty universal set. Then, we can define a Generalized Fuzzy Set in X by the expression G={<x, $\mu_G(x)$, $\nu_G(x)$>x ϵ X}; where the functions $\mu_G(x):X\rightarrow [0,1]$ denotes the degree of membership (support for membership) and $\nu_G(x):X\rightarrow[0,1]$ denotes degree of non-membership (support against membership) of the element x ϵ X to the set G such that it satisfies the condition $0 \leq (\mu_G(x))^n + (\nu_G(x))^m \leq 1$ (n, m are non-negative integers).

The degree of indeterminacy of x ϵ X to G is defined by $\pi_G(x) = [1-\{(\mu_G(x))^n + (\nu_G(x))^m\}]^{(1/\mathrm{lcm}(n,\ m))}$ and $\pi_G(x) \in [0, 1]$ satisfying the condition $(\mu_G(x))^n + (\nu_G(x))^m + (\pi_G(x))^{\mathrm{lcm}(n,\ m)} = 1$.

Definition (Hyperbolic Fuzzy Set): [2] Let us consider X as a non-empty universal set. Then, we can define an Hyperbolic Fuzzy Set in X by the expression H={<x, $\mu_H(x)$, $\nu_H(x)$>x ϵ X}; where the functions $\mu_H(x):X\rightarrow [0,1]$ denotes the degree of optimism and $\nu_H(x):X\rightarrow[0,1]$ denotes the degree of pessimism of the element x ϵ X to the set H such that it satisfies the condition $0 \leq (\mu_H(x))\times (\nu_H(x)) \leq 1$.

3. NECESSITY FOR A BETTER ALTERNATIVE TO EXISTING FORMS

The concept of the Fuzzy Set proposed by Zadeh just consisted of a single parameter, the membership degree (μ), and so it is not sufficient to solve problems related to uncertainty.

The subsequent development occurred in the form of Intuitionistic Fuzzy Set (IFS) by Attanosov. He introduced the concept of non-membership degree (ν) along with the membership degree (μ) such that it satisfies the condition $\mu+\nu\leq 1$. However, IFS fails when $\mu+\nu> 1$. Yager then introduced the concept of Pythagorean Fuzzy Set (PFS) such that it satisfies the condition $\mu^2+\nu^2\leq 1$. However, it also fails when $\mu^2+\nu^2>1$. Similarly, Senapati and Yager introduced the concept of Fermatean Fuzzy Set (FFS) such that it satisfies the condition $\mu^3+\nu^3\leq 1$.

Yager again generalized the concept of IFS, PFS and FFS in the form of q-Rung Orthopair Fuzzy Set (q-ROFS) by adding qth times the sum of the membership and the non-membership degree that it satisfies the condition $\mu^q+\nu^q\leq 1$. It covers nearly all the membership and non-membership values.

A further generalized concept of n, m - Rung Orthopair Fuzzy Set (n, m-ROFS) and Generalized Fuzzy Set (GFS) were conceptualized by adding n^{th} times the sum of the membership and the m^{th} times the non-membership degree that it satisfies the condition $\mu^n+\nu^m\leq 1$. It covers nearly all the membership and non-membership values.

However, taking inspiration from Two Factor Theory, considering the membership degree (or the non-membership degree) as 1 and the non-membership (or the membership degree) as non-zero, q-ROFS, GFS and n, m-ROFS fails. We can see that in the following figures.

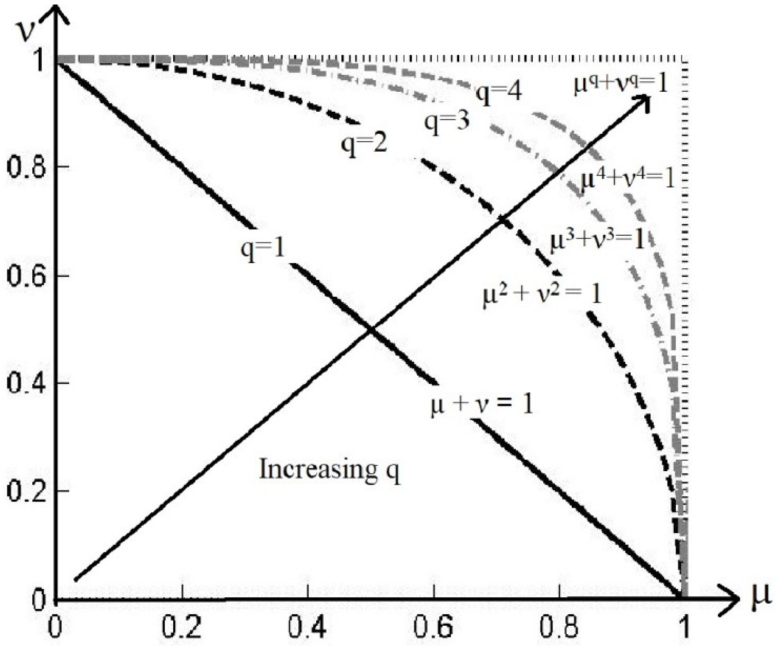


Figure 1 : Geometric Interpretation of q-ROFS

In the Figure 1, we consider the membership degree and the non-membership degree as its axes. We consider the points under the graph $\mu^q+\nu^q\leq1$. When we consider q=1, then it becomes the condition for IFS, $\mu+\nu\leq1$. Therefore, considering the values μ and ν such that it satisfies the IFS condition, it will lie inside the line $\mu+\nu=1$. However, when we consider the values of μ and ν such that $\mu+\nu>1$, it will lie outside $\mu+\nu=1$, then IFS fails.

Considering, q=2, it becomes the condition for PFS, $\mu^2+\nu^2\leq1$. Hence, the values of μ and ν satisfying the PFS condition will lie inside the curve $\mu^2+\nu^2=1$. However, when the values of μ and ν are considered such that $\mu^2+\nu^2>1$, it will lie outside the curve $\mu^2+\nu^2=1$, then PFS fails.

Again, for, q=3, it becomes the condition for FFS, $\mu^3+\nu^3\leq1$. Hence, the values of μ and ν satisfying the FFS condition will lie inside the curve $\mu^3+\nu^3=1$. However, when the values of μ and ν are considered such that $\mu^3+\nu^3>1$, it will lie outside the curve $\mu^3+\nu^3=1$, then FFS fails.

Therefore, we can see that on increasing the values of q, the range of consideration of the values of μ and ν increases, nearly representing any possible values for membership degree and non-membership degree.

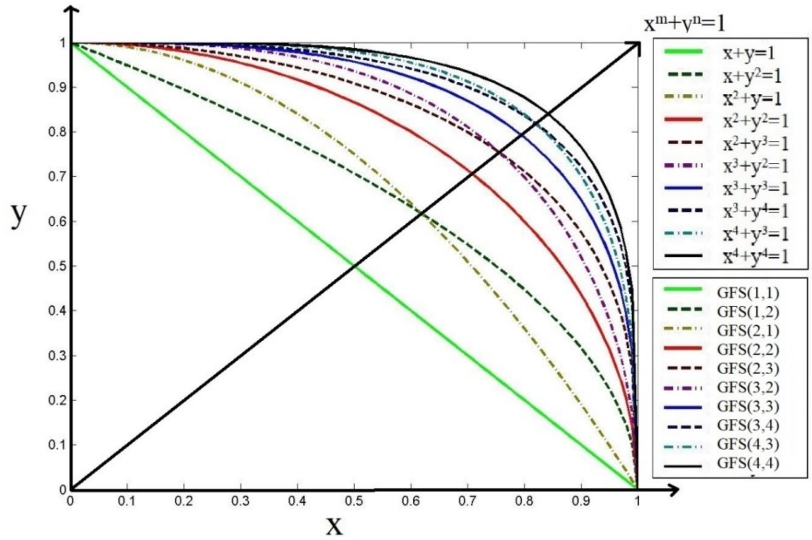
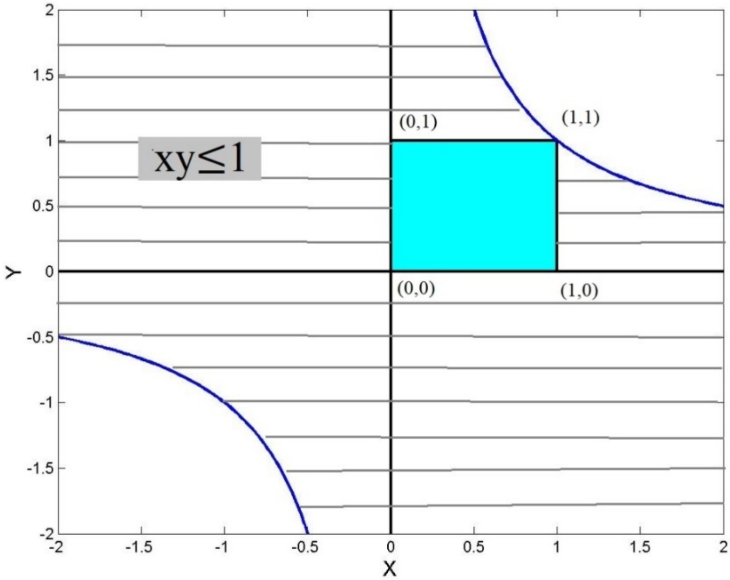


Figure 2: Geometric Interpretation of GFS

The GFS serves as an extension of q-ROFS which itself is an extension of IFS, PFS and FFS. In the Figure 2, we can see that GFS (1, 1) which is IFS provides the least range for consideration of membership degree and non-membership degree. GFS (2, 2) which is PFS provides a greater range than IFS for consideration of membership degree and non-membership degree. But, in between GFS (1, 1) and GFS (2, 2), there lies GFS (1, 2) and GFS (2, 1) which provides a greater range of consideration of membership degree and non-membership degree values in comparison to GFS (2, 2). Similarly, GFS (2, 3) and GFS (3, 2) provides a greater range of membership degree and non-membership degree values in comparison to PFS but lesser than GFS (3, 3) which is FFS. Finally, the range of GFS (3, 4) and GFS (4, 3) lies between GFS (3, 3) and GFS (4, 4).

However, it fails when the membership degree is 1 and the non-membership degree is non-zero and vice-versa (i.e., $\mu=1$ and $\nu\neq0$ and vice-versa).

Thus, we need a better alternative to consider membership degree and non-membership degree as 1, thus giving more independence in selecting the membership degree and the non-membership degree. Here comes the role of Hyperbolic Fuzzy Set, where we can consider degree of optimism as 1 and a non-zero degree of pessimism and vice-versa (i.e., $\mu=1$ and $\nu\neq0$ and vice-versa). The geometrical interpretation of HFS in given in Figure 3. In the Figure, the colored square rectangular region shows the range of consideration of HFS values. We can see that even the degree of optimism as 1 and degree of pessimism as 1 could be considered in HFS which is not possible in the existing forms of q-ROFS, GFS and n, m-ROFS.



.

Figure 3: Geometric Interpretation of HFS

For a better understanding, we consider an example as shown below:

Example: Let, A ϵ IFS. Suppose $\mu_A = 0.6$, $\nu_A = 0.5$.

Clearly, we can see that (0.6) + (0.5) > 1. So IFS fails here.

However, if we consider A ϵ PFS, then, $(0.6)^2 + (0.5)^2 < 1$. So, PFS succeeds here.

Also, if A ϵ GFS (2, 1), GFS (1, 2), then, $(0.6)^2 + (0.5) < 1$ and $(0.6) + (0.5)^2 < 1$. So, GFS (1, 2), GFS (2, 1) succeeds here.

Now, for A ϵ PFS. Suppose $\mu_A = 0.8$, $\nu_A = 0.7$.

Clearly, we can see that $(0.8)^2 + (0.7)^2 > 1$. So, PFS fails here.

Further, $(0.8)^2 + (0.7) > 1$ and $(0.8) + (0.7)^2 > 1$. So, GFS (2, 1), GFS (1, 2) fails here.

However, if A ϵ q-ROFS. Then, $(0.8)^q + (0.7)^q < 1$, for $q > 3$. So, q-ROFS (FFS for q=3) succeeds here.

Also, if A ϵ GFS (3, 2), $(0.8)^3 + (0.7)^2 > 1$. So, GFS (3, 2) fails here.

But, if A ϵ GFS (2, 3), $(0.8)^2 + (0.7)^3 > 1$. So, GFS (2, 3) succeeds here.

Let us consider, A ϵ q-ROFS, GFS. Suppose $\mu_A=1.0$, $\nu_A=0.1$.

Clearly, $(1.0)^q + (0.1)^q > 1$, for any q. So, q-ROFS fails here.

Also, $(1.0)^n + (0.1)^m > 1$, for any n, m. So, GFS fails here.

But, for, A ϵ HFS. Suppose $\mu_A=1.0$, $\nu_A=0.1$.

Clearly, $(1.0) \times (0.1) < 1$. So, HFS succeeds here.

In the example we can see that HFS serves as a better concept w.r.t the existing forms.

4. CONCLUSION

From the study, the various approaches of development of Fuzzy Set Theory have been discussed. With the gradual developments, the independence of the membership degree and the non-membership degree has increased. The role of q-ROFS in decision making is quite remarkable due to its wider range of consideration of membership degree and non-membership degree values in comparison to IFS, PFS and FFS as it serves as their generalization such that for q=1, q-ROFS becomes IFS, PFS for q=2, and FFS for q=3 and also, we could consider higher powers of q as per convenience. But, GFS (and n, m-ROFS) serves as a further extension of q-ROFS such that IFS<PFS<FFS<q-ROFS<GFS. Thus, GFS (1, 1) is IFS; GFS (2, 2) is PFS; GFS (3, 3) is FFS and finally GFS (q, q) is q-ROFS. The recent formulation of HFS serves as the best alternative where the independence in consideration of the values in maximum. New methods could be developed based on HFS aiding in decision making problems.

References

[1] Atanassov, K. T., & Stoeva, S. (1986). Intuitionistic fuzzy sets. *Fuzzy sets and Systems*, *20*(1), 87-96.

[2] Dutta, P., & Borah, G. (2023). Construction of Hyperbolic Fuzzy Set and its Applications in Diverse COVID-19 Associated Problems. *New Mathematics and Natural Computation*, *19*(01), 217-288.

[3] Herzberg, F. I. (1966). Work and the Nature of Man.

[4] Ibrahim, H. Z., & Alshammari, I. (2022). n, m-Rung Orthopair Fuzzy Sets With Applications to Multicriteria Decision Making. *IEEE Access*, *10*, 99562-99572.

[5] Munir, M., Kausar, N., & Khan, S. I. (2023). Generalized fuzzy sets and their applications in purchase satisfaction, personnel posting and disease diagnosis. *Soft Computing*, *27*(7), 3907-3920.

[6] Senapati, T., & Yager, R. R. (2020). Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, *11*, 663-674.

[7] Yager, R. R. (2013). Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on fuzzy systems*, *22*(4), 958-965.

[8] Yager, R. R. (2016). Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, *25*(5), 1222-1230.

[9] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.