A NEW EXTENSION OF THE MARSHALL-OLKIN KUMARASWAMY-G FAMILY OF DISTRIBUTIONS: ITS PROPERTIES AND APPLICATIONS WITH FAILURE TIME DATA

Laba Handique

Department of Statistics, Darrang College, Tezpur-784001, Assam, India

Corresponding Author Email: handiquelaba@gmail.com

Abstract

This paper introduces a new family of continuous probability distributions called the Marshall-Olkin Kumaraswamy-G Poisson family of distribution. Some of its mathematical properties including explicit expressions for the order statistics, probability weighted moments, moments generating function, mean deviation and Rényi entropy are derived. The estimation of the model parameters is performed by the maximum likelihood method. The flexibility of the proposed family is illustrated by means of one real life application to failure time data set.

***Key words*:** MO-G distribution, Kw-G distribution, MLE, AIC, A, W

1. INTRODUCTION

Generating new distributions starting with a base line distribution by adding one or more additional parameters through various mechanisms is an area of research in the field of the probability distribution which have seen lot of work of late. The basic motivation of this paper is to bring in more flexibility in the modelling failure type of data generated from real life situation. Extension of existing well-known distributions to enhance flexibility in modelling variety of data has attracted attention of researchers recently. Some of the notable new family of distributions proposed of late includes among others the Poisson-G family (Abouelmagd *et al*., 2017), Marshall-Olkin Kumaraswamy-G family (Handique *et al*., 2017), beta Kumaraswamy-G family (Handique *et al*., 2017), beta generated Kumaraswamy Marshall-Olkin-G family (Handique and Chakraborty, 2017), generalized Burr XII family (Handique and Chakraborty, 2018), exponentiated generalized-G Poisson family (Gokarna and Haitham, 2018), beta-G Poisson family (Gokarna *et al*., 2019), exponentiated generalized Marshall-Olkin family (Handique *et al*., 2019), zero truncated Poisson family (Abouelmagd *et al*., 2019), Generalized Modified exponential-G family (Handique *et al*., 2020), Poisson Transmuted-G family (Handique *et al*., 2021), beta generalized Marshall-Olkin-G family (Handique *et al*., 2021), Odd Half-Cauchy family (Chakraborty *et al*., 2021), McDonald Lindley-Poisson family (Percontini *et al*., 2021), Kumaraswamy Poisson-G family (Chakraborty *et al*., 2022), Beta Poisson-G family (Handique *et al*., 2022), generalized odd linear exponential family (Farrukh *et al*., 2022) and complementary geometric-Topp-Leone-G family (Handique *et al*., 2023) among others.

Here briefly introduce the Marshall Olkin-G (MO-G) family (Marshall and Olkin, 1997), Kumaraswamy-G (Kw-G) (Cordeiro and de Castro, 2011) family and Marshall Olkin Kumaraswamy-G (Handique et al., 2017) family of distributions.

1.1 Marshall-Olkin-G (MO-G) family of distributions

Starting with a given baseline distribution with probability density function (pdf) $g(x)$ and cumulative distribution function (cdf) $G(x)$ Marshall and Olkin (1997) introduced a new family of distributions with sf $\bar{F}^{\text{MOG}}(x;\alpha)$ by introducing an extra parameter $\alpha > 0$. The survival function (sf) $\bar{F}^{\text{MOG}}(x;\alpha)$ of the $\text{MOG}(\alpha)$ family of distributions is defined by

$$\bar{F}^{\text{MOG}}(x;\alpha) = \frac{\alpha\,\bar{G}(x)}{1-\bar{\alpha}\,\bar{G}(x)}$$

where, $-\infty < x < \infty$, $\alpha > 0$ and $\bar{\alpha} = 1-\alpha$. Now the cdf and pdf of the $\text{MOG}(\alpha)$ family of distributions is given by

$$F^{\text{MOG}}(x;\alpha) = \frac{G(x)}{1-\bar{\alpha}\,\bar{G}(x)} \quad \text{and} \quad f^{\text{MOG}}(x;\alpha) = \frac{\alpha\, g(x)}{[1-\bar{\alpha}\,\bar{G}(x)]^2}.$$

where $g(x)$ and $G(x)$ is the pdf and cdf of the baseline distribution. If $\alpha = 1$, then $\bar{F}^{\text{MOG}}(x;\alpha) = \bar{G}(x)$.

1.2 Kumaraswamy-G (Kw-G) family of distributions

For a baseline cdf $G(x)$ with pdf $g(x)$, Cordeiro and de Castro (2011) defined $\text{KwG}(a,b)$ distribution with sf, cdf and pdf are

$$\bar{F}^{\text{KwG}}(x;a,b) = [1-G(x)^a]^b,\ F^{\text{KwG}}(x;a,b) = 1-[1-G(x)^a]^b$$

and
$$f^{\text{KwG}}(x;a,b) = a\,b\,g(x)\,G(x)^{a-1}[1-G(x)^a]^{b-1}.$$

where $x > 0,\ g(x) = G'(x)$ and $a > 0,\ b > 0$ are additional shape parameters besides those of the baseline distribution which influence the skewness and tail weights.

1.3 Marshall-Olkin Kumaraswamy-G (MOKw-G) family of distributions

Handique *et al*. (2017) proposed a new extension of the $\text{MOG}(\alpha)$ family by considering the cdf and pdf of $\text{KwG}(a,b)$ distribution in the $\text{MOG}(\alpha)$ formulation and call it $\mathrm{MOKw-G}(\alpha, a, b)$ distribution with pdf is given by

$$f^{\mathrm{MOKwG}}(x;\alpha,a,b)=\frac{\alpha\, a\, b\, g(x)\, G(x)^{a-1}\,[1-G(x)^{a}\,]^{b-1}}{[1-\bar{\alpha}\ \{1-G(x)^{a}\}^{b}\ ]^{2}} \quad (1)$$

$$,0<x<\infty,\ \alpha>0, a>0,\ b>0$$

By using equation (1) the corresponding cdf, sf and hrf of $\mathrm{MOKw-G}(\alpha, a, b,)$ are respectively obtained as

$$F^{\mathrm{MOKwG}}(x;\alpha,a,b)=\frac{1-[1-G(x)^{a}]^{b}}{1-\bar{\alpha}[1-G(x)^{a}\,]^{b}} \quad (2)$$

$$\bar{F}^{\mathrm{MOKwG}}(x;\alpha,a,b)=\frac{\alpha[1-G(x)^{a}]^{b}}{1-\bar{\alpha}[1-G(x)^{a}\,]^{b}} \quad (3)$$

$$h^{\mathrm{MOKwG}}(x;\alpha,a,b)=\frac{a\, b\, g(x)\, G(x)^{a-1}[1-G(x)^{a}]^{-1}}{1-\bar{\alpha}[1-G(x)^{a}]^{b}} \quad (4)$$

The pdf in equation (1) for $\alpha=1$, reduces to that of $\mathrm{KwG}(a,b)$ and for $a=b=1$, reduces to that of $\mathrm{MOG}(\alpha)$.

2. THE PROPOSED MODEL

Suppose that the failure time of each subsystem follows the $\mathrm{MOKwG}(\alpha, a, b)$ distribution above. Let Y_i denote the failure time of the i^{th} subsystem and *X* denote the time to failure of the first out of the *N* functioning subsystems that is $X=\min\{Y_1, Y_2, \ldots, Y_n\}$. Then the conditional cdf of *X* given *N* is

$$F(x;\alpha,a,b/N)=1-\Pr(X>x/N)=1-P(Y_i>x)^{N}=1-[1-G^{\mathrm{MOKwG}}(x;\alpha,a,b)]^{N}$$

So, the unconditional cdf of *X* (for $x>0$) can be expressed as

$$F^{MOKw-GP}(x;\alpha,a,b,\lambda)=\frac{1}{(e^{\lambda}-1)}\sum_{n=1}^{\infty}\frac{\lambda^{n}\,1-[1-G^{\mathrm{MOKwG}}(x;\alpha,a,b)]^{n}}{n!}$$

$$= \frac{1-\exp[-\lambda G^{\mathrm{MOKwG}}(x;\alpha,a,b)]}{1-e^{-\lambda}} \tag{5}$$

Note here that if we take $X = \max\{Y_1, Y_2, ..., Y_n\}$ and proceed as above the new cdf happens to be

$$F^{MOKw-GP}(x;\alpha,a,b,\lambda) = \frac{\exp[\lambda G^{\mathrm{MOKwG}}(x;\alpha,a,b)]-1}{e^{\lambda}-1} \tag{6}$$

It is easy to combine (5) and (6), as a new family of distributions with cdf

$$F^{MOKw-GP}(x;\alpha,a,b,\lambda) = \frac{1-\exp[-\lambda G^{\mathrm{MOKwG}}(x;\alpha,a,b)]}{1-e^{-\lambda}},\ \lambda \in R-\{0\} \tag{7}$$

Above refer to the distribution in equation (7) as the Marshall-Olkin Kumaraswamy-G Poisson family (**"MOKw-GP"** in short) of distribution. The corresponding pdf and hrf of MOKw-GP(α,a,b,λ) family is given by

$$f^{MOKw-GP}(x;\alpha,a,b,\lambda) = (1-e^{-\lambda})^{-1}\lambda\, g^{\mathrm{MOKwG}}(x;\alpha,a,b)\exp[-\lambda G^{\mathrm{MOKwG}}(x;\alpha,a,b)] \tag{8}$$

and $$h^{MOKw-GP}(x;\alpha,a,b,\lambda) = \frac{\lambda\, g^{\mathrm{MOKwG}}(x;\alpha,a,b)\exp[-\lambda G^{\mathrm{MOKwG}}(x;\alpha,a,b)]}{\exp[-\lambda G^{\mathrm{MOKwG}}(x;\alpha,a,b)]-e^{-\lambda}}$$

$$,\quad \lambda \in R-\{0\};\ -\infty < x < \infty$$

The main advantage of the proposed family of distribution appears to be its enhanced flexibility. Moreover distributions from this extended family is expected show significant improvement in data adjustment when compared to it sub models and other existing ones with respect to various model selection criteria, test of goodness-of-fits.

In the present work concentrate on the Marshall-Olkin Kumaraswamy exponential Poisson (MOKw-EP) distribution a particular distribution of the proposed family which is derived by considering $g(x)=\beta e^{-\beta x}$ and $G(x)=1-e^{-\beta x}$, $x>0$, $\beta>0$ in $\text{MOKw-EP}(\alpha,a,b,\beta,\lambda)$. The pdf, cdf and sf respectively of the derived distribution are respectively given as

$$f^{MOKw-EP}(x;\alpha,a,b,\beta,\lambda)=(1-e^{-\lambda})^{-1}\lambda\frac{\alpha\,a\,b\,\beta\,e^{-\beta x}(1-e^{-\beta x})^{a-1}[1-(1-e^{-\beta x})^{a}]^{b-1}}{[1-\bar{\alpha}\,\{1-(1-e^{-\beta x})^{a}\}^{b}\,]^{2}}$$

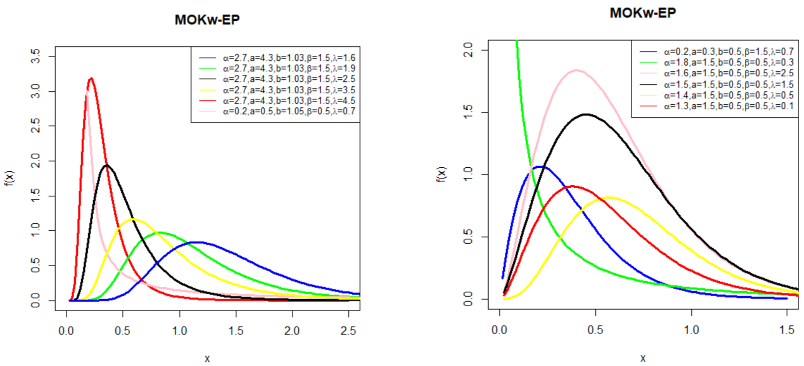
$$\times\exp\left[-\lambda\frac{1-[1-(1-e^{-\beta x})^{a}]^{b}}{1-\bar{\alpha}[1-(1-e^{-\beta x})^{a}]^{b}}\right]$$

$$F^{MOKw-EP}(x;\alpha,a,b,\beta,\lambda)=(1-e^{-\lambda})^{-1}\left[1-\exp\left[-\lambda\frac{1-[1-(1-e^{-\beta x})^{a}]^{b}}{1-\bar{\alpha}[1-(1-e^{-\beta x})^{a}]^{b}}\right]\right]$$

and

$$\bar{F}^{MOKw-EP}(x;\alpha,a,b,\beta,\lambda)=1-\frac{1}{1-e^{-\lambda}}\left[1-\exp\left[-\lambda\frac{1-[1-(1-e^{-\beta x})^{a}]^{b}}{1-\bar{\alpha}[1-(1-e^{-\beta x})^{a}]^{b}}\right]\right]$$

Now the plots of the pdf of the $\text{MOKw-EP}(\alpha,a,b,\beta,\lambda)$ for some selected values of parameters to in Figure 1 to check the shapes assumed the distribution



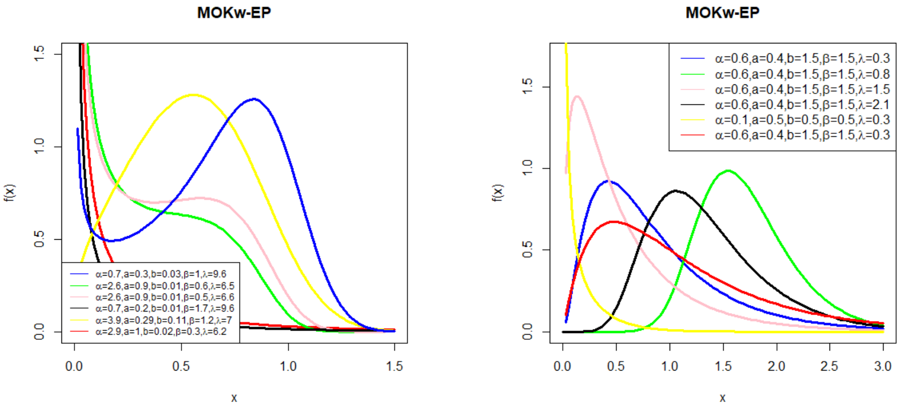


Figure 1: Density plots of the $\text{MOKw-EP}(\alpha, a, b, \beta, \lambda)$ distribution

From the plots of the pdf of the $\text{MOKw-EP}(\alpha, a, b, \beta, \lambda)$ in Figure 1 for different parameter values it can be seen that the distribution is very flexible and can offer different types of shapes of density right skewed, left skewed, high/ low peak and symmetric as well.

The rest of the article is outlined as follows. In Section 3, derive a very useful representation for the MOKw-GP density and distribution function also obtain some general mathematical properties of the proposed family including order statistics, probability weighted moments, moment generating function, mean deviation and Rényi entropy. Maximum likelihood estimation of the model parameters is investigated in Section 4. In Section 5, one application to failure time data set to illustrate the potentiality of some special models of the proposed family. Finally, concluding remarks are presented in Section 6.

3. EXPANSIONS OF THE PDF AND CDF

Here express (7) and (8) as infinite series expansion to show that the $\text{MOKw-GP}(\alpha, a, b, \lambda)$ can be written as a linear combination of

MOKw-G(α, a, b) distributions. These expressions will be helpful to study the mathematical and statistical characteristics of the MOKw-GP(α, a, b, λ) family.

Using the power series for the exponential function, in equation (8) as

$$f^{\text{MOKwGP}}(x;\alpha,a,b,\lambda) = g^{\text{MOKwG}}(x;\alpha,a,b)\sum_{i=0}^{\infty}\vartheta_i\,[G^{\text{MOKwG}}(x;\alpha,a,b)]^i \tag{9}$$

$$=\sum_{i=0}^{\infty}\vartheta'_i\,\frac{d}{dx}[G^{\text{MOKwG}}(x;\alpha,a,b]^{i+1} \tag{10}$$

where $\vartheta'_i = \dfrac{(-1)^i\,\lambda^{i+1}}{(1-e^{-\lambda})\,(i+1)i!}$ and $\vartheta_i = \vartheta'_i\,(i+1)$

Using Taylor series expansion cdf of (7) as

$$F^{\text{MOKwGP}}(x;\alpha,a,b,\lambda) = \sum_{j=0}^{\infty}\mu_j\,[G^{\text{MOKwG}}(x;\alpha,a,b)]^j \tag{11}$$

where $\mu_j = \dfrac{(-1)^{j+1}\,\lambda^j}{(1-e^{-\lambda})\,j!}$

3.1 Distribution of Order Statistics

Consider a random sample $X_1, X_2, \ldots, X_n$ from any MOKw-GP(α, a, b, λ) distribution. Let $X_{r:n}$ denote the r^{th} order statistic. The pdf of $X_{r:n}$ can be expressed as

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} f^{\text{MOKw-GP}}(x;\alpha,a,b,\lambda) F^{\text{MOKw-GP}}(x;\alpha,a,b,\lambda)^{r-1}\{1-F^{\text{MOKw-GP}}(x;\alpha,a,b,\lambda)\}^{n-r}$$

$$= \frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m} f^{\text{MOKw-GP}}(x;\alpha,a,b,\lambda)\{F^{\text{MOKw-GP}}(x;\alpha,a,b,\lambda)\}^{m+r-1}$$

The pdf of the r^{th} order statistic for of the MOKw-GP(α,a,b,λ) can be derived by using the expansion of the pdf and cdf as

$$f_{r:n}(x)=\frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\sum_{i=0}^{\infty}\vartheta_i[G^{\text{MOKwG}}(x;\alpha,a,b)]^i\,g^{\text{MOKwG}}(x;\alpha,a,b)$$

$$\times[\sum_{j=0}^{\infty}\mu_j[G^{\text{MOKwG}}(x;\alpha,a,b]^j]^{m+r-1}$$, where ϑ_i and μ_j are defined above.

Using power series raised to power for positive integer $n\,(\geq 1)$, $\left(\sum_{i=0}^{\infty}a_i u^i\right)^n=\sum_{i=0}^{\infty}c_{n,i}u^i$, where the coefficient $c_{n,i}$ for $i=1,2,...$ are easily obtained

from the recurrence equation $c_{n,i}=(i\,a_0)^{-1}\sum_{m=1}^{i}[m(n+1)-i]a_m c_{n,i-m}$ where $c_{n,0}=a_o^n$.

Now $$\left[\sum_{j=0}^{\infty}\mu_j[G^{\text{MOKwG}}(x;\alpha,a,b]^j\right]^{m+r-1}=\sum_{j=0}^{\infty}d_{m+r-1,j}[G^{\text{MOKwG}}(x;\alpha,a,b)]^j$$

Therefore the density function of the r^{th} order statistics of MOKw-GP(α,a,b,λ) distribution can be expressed as

$$f_{r:n}(x)$$
$$=\frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\vartheta_i\,d_{m+r-1,j}[G^{\text{MOKwG}}(x;\alpha,a,b)]^{i+j}\,g^{\text{MOKwG}}(x;\alpha,a,b)$$

$$=\sum_{i,j=0}^{\infty}\kappa_{ij}[G^{\text{MOKwG}}(x;\alpha,a,b)]^{i+j}\,g^{\text{MOKwG}}(x;\alpha,a,b) \qquad (12)$$

$$=\sum_{i,j=0}^{\infty}\frac{\kappa_{ij}}{(i+j+1)}\frac{d}{dx}[G^{\text{MOKwG}}(x;\alpha,a,b)]^{i+j+1}$$

where $\kappa_{ij}=\frac{n!}{(r-1)!(n-r)!}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}\vartheta_i\,d_{m+r-1,j}$

3.2 Probability Weighted Moments

The probability weighted moments (PWM), first proposed by Greenwood *et al.* (1979), are expectations of certain functions of a random variable whose mean exists. The $(p,q,r)^{th}$ PWM of T is defined by $\Gamma_{p,q,r} = \int_{-\infty}^{\infty} x^p F(x)^q \, [1-F(x)]^r \, f(x)\, dx \cdot$

From equation (9) the s^{th} moment of T can be written as

$$E(X^s) = \int_{-\infty}^{\infty} x^s \; f^{\text{MOKwGP}}(x;\alpha,a,b,\lambda)\; dx$$

$$= \sum_{i=0}^{\infty} \vartheta_i \int_{-\infty}^{\infty} x^s \; [G^{\text{MOKwG}}(x;\alpha,a,b)]^i \, g^{\text{MOKwG}}(x;\alpha,a,b)\, dx$$

$$= \sum_{i=0}^{\infty} \vartheta_i \; \Gamma_{s,\,i,0}$$

where $\Gamma_{p,q,r} = \int_{-\infty}^{\infty} x^p \; [F^{\text{MOKwG}}(x;\alpha,a,b)]^q \{1-F^{\text{MOKwG}}(x;\alpha,a,b)\}^r \, f^{\text{MOKwG}}(x;\alpha,a,b)\, dx$ is the PWM of MOKwG(α,a,b) distribution. Therefore the moments of the MOKwGP(α,a,b,λ) may be expressed in terms of the PWMs of MOKwG(α,a,b) .

Proceeding similarly can express s^{th} moment of the r^{th} order statistic $X_{r:n}$ in a random sample of size n from MOKwGP(α,a,b,λ) on using equation (12) as $E(X^s_{r;n}) = \sum_{i,j=0}^{\infty} \kappa_{ij} \; \Gamma_{s,\,i+j,\,0}$, where κ_{ij} defined in above.

3.3 Moment Generating Function

The moment generating function of MOKw-GP(α,a,b,λ) family can be easily expressed in terms of those of the exponentiated MOKwG(α,a,b) distribution using the results of Section 3. For example using equation (10) it can be seen that

$$M_X(s)=E[e^{sX}]=\int_{-\infty}^{\infty} e^{sx} f^{\text{MOKwGP}}(x;\alpha,a,b,\lambda)\ dx=\int_{-\infty}^{\infty} e^{sx}\sum_{i=0}^{\infty}\vartheta_i' \frac{d}{dx}[G^{\text{MOKwG}}(x;\alpha,a,b]^{i+1}\ dx$$

$$=\sum_{i=0}^{\infty}\vartheta_i'\int_{-\infty}^{\infty} e^{sx}\frac{d}{dx}[G^{\text{MOKwG}}(x;\alpha,a,b)]^{i+1}\ dx\ =\sum_{i=0}^{\infty}\vartheta_i M_X(s)$$

where $M_X(s)$ is the mgf of a exponentiated MOKwG(α,a,b) distribution.

3.4 Mean Deviation

Let X be the MOKwGP(α,a,b,λ) random variable with mean $\mu=E(X)$ and median $M=\text{Median}\ (X)=Q(0.5)$. The mean deviation from the mean $[\delta_\mu(X)=E(|X-\mu|)]$ and the mean deviation from the median $[\delta_M(X)=E(|X-M|)]$ can be expressed as

$$\delta_\mu(X)=\int_{-\infty}^{\infty}|X-\mu|\ f(x)\,dx=\int_{-\infty}^{\mu}(\mu-x)\ f(x)\,dx+\int_{\mu}^{\infty}(x-\mu)\ f(x)\,dx=2\mu F(\mu)-2\Psi(\mu)$$

and

$$\delta_M(X)=\int_{-\infty}^{\infty}|X-M|\ f(x)\,dx=\int_{-\infty}^{M}(M-x)\ f(x)\,dx+\int_{M}^{\infty}(x-M)\ f(x)\,dx=\mu-2\Psi(M)$$

respectively, where $F(\cdot)$ is the cdf of the MOKwGP(α,a,b,λ) distribution, and

$\Psi(t)=\int_{-\infty}^{t} x\,f(x)\,dx$ where $\Psi(t)$ as follows:

$\Psi(t)=\sum_{i=0}^{\infty}\vartheta_i\int_{-\infty}^{t} x[G^{\text{MOKwG}}(x;\alpha,a,b)]^i\, g^{\text{MOKwG}}(x;\alpha,a,b)\,dx$, where ϑ_i defined in Section 3.

3.5 Rényi entropy

The Rényi entropy is defined by $I_R(\vartheta) = (1-\vartheta)^{-1}\log\left(\int_{-\infty}^{\infty} f(x)^{\vartheta}dx\right)$, where $\vartheta > 0$ and $\vartheta \neq 1$. Using power series exponential function in equation (9) can write as

$$f^{\mathrm{MOKwGP}}(x;\alpha,a,b,\lambda)^{\vartheta} = g^{\mathrm{MOKwG}}(x;\alpha,a,b)^{\vartheta}\sum_{m=0}^{\infty}\eta_m[G^{\mathrm{MOKwG}}(x;\alpha,a,b)]^{i\vartheta}$$

Thus $$I_R(\vartheta) = (1-\vartheta)^{-1}\log\left(\int_0^{\infty} g^{\mathrm{MOKwG}}(x;\alpha,a,b)^{\vartheta}\sum_{m=0}^{\infty}\eta_m[G^{\mathrm{MOKwG}}(x;\alpha,a,b)]^{i\vartheta}dx\right)$$

$$= (1-\vartheta)^{-1}\log\left(\sum_{m=0}^{\infty}\eta_m\int_0^{\infty} g^{\mathrm{MOKwG}}(x;\alpha,a,b)^{\vartheta}[G^{\mathrm{MOKwG}}(x;\alpha,a,b)]^{i\vartheta}dx\right)$$

where $\eta_m = \dfrac{(-1)^m\lambda^{\vartheta(i+1)}}{(1-e^{-\lambda})^{\vartheta}\ m!}$

4. MAXIMUM LIKELIHOOD ESTIMATION METHOD:

This section is devoted to the estimation of the $\mathrm{MOKwGP}(\alpha,a,b,\lambda)$ model parameters via the maximum likelihood (ML) method.

Let $x = (x_1, x_2, \ldots, x_n)$ be a random sample of size n from $\mathrm{MOKwGP}(\alpha,a,b,\lambda)$ with parameter vector $\boldsymbol{\rho} = (\alpha,a,b,\lambda,\xi)$, where $\boldsymbol{\xi} = (\xi_1,\xi_2,\ldots,\xi_q)$ is the parameter vector of *G*. The log-likelihood function is written as

$$\ell \quad \ell \qquad \text{og}(\alpha ab\lambda) - n\log(1-e^{-\lambda}) + \sum_{i=1}^{n}\log[g(x_i,\boldsymbol{\xi})] + (a-1)\sum_{i=1}^{n}\log[G(x_i,\boldsymbol{\xi})]$$

$$+(b-1)\sum_{i=1}^{n}\log[1-G(x_i,\boldsymbol{\xi})^a] - 2\sum_{i=1}^{n}([1-\bar{\alpha}\{1-G(x_i,\boldsymbol{\xi})^a\}^b] - \lambda\sum_{i=1}^{n}\left[\frac{1-[1-G(x;\boldsymbol{\xi})^a]^b}{1-\bar{\alpha}[1-G(x;\boldsymbol{\xi})^a]^b}\right]$$

This log-likelihood function can not be solved analytically because of its complex form but it can be maximized numerically by employing global optimization methods available with the **software's R**. By taking the partial derivatives of the log-likelihood function with respect to α, a, b and λ obtain the components of the score vector $U_{\boldsymbol{\rho}} = (U_\alpha, U_a, U_b, U_\lambda, U_\xi)$.

The asymptotic variance-covariance matrix of the MLEs of parameters can obtained by inverting the Fisher information matrix $\mathrm{I}(\boldsymbol{\rho})$ which can be derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The $i\,j^{th}$ elements of $\mathrm{I}_n(\boldsymbol{\rho})$ are given by

$$\mathrm{I}_{i\,j} = -E[\partial^2 l(\boldsymbol{\rho}) / \partial \rho_i \partial \rho_j] , \quad i, j = 1, 2, 3, 4+q .$$

The exact evaluation of the above expectations may be cumbersome. In practice one can estimate $\mathrm{I}_n(\boldsymbol{\rho})$ by the observed Fisher's information matrix $\hat{\mathrm{I}}_n(\hat{\boldsymbol{\rho}}) = (\hat{\mathrm{I}}_{i\,j})$ defined as

$$\hat{\mathrm{I}}_{i\,j} \approx \left(-\partial^2 l(\boldsymbol{\rho}) / \partial \rho_i \partial \rho_j\right)_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}} , \quad i, j = 1, 2, 3, 4+q .$$

Using the general theory of MLEs under some regularity conditions on the parameters as $n \to \infty$ the asymptotic distribution of $\sqrt{n}\,(\hat{\boldsymbol{\rho}} - \boldsymbol{\rho})$ is $N_k(0, V_n)$ where $V_n = (v_{jj}) = \mathrm{I}_n^{-1}(\boldsymbol{\rho})$. The asymptotic behaviour remains valid if V_n is replaced by $\hat{V}_n = \hat{\mathrm{I}}^{-1}(\hat{\boldsymbol{\rho}})$. Using this result large sample standard errors of j^{th} parameter ρ_j is given by $\sqrt{\hat{v}_{j\,j}}$.

5. REAL LIFE APPLICATION FOR FAILURE TIME DATA

Here consider fitting of one failure time data set to show that the distributions from the proposed MOKw-EP distribution can provide better model than the corresponding distributions exponential (Exp), moment exponential (ME), Marshall-Olkin exponential (MO-E) (Marshall and Olkin, 1997), generalized Marshall-Olkin

exponential (GMO-E) (Jayakumar and Mathew, 2008), Kumaraswamy exponential (Kw-E) (Cordeiro and de Castro, 2011), Beta exponential (BE) (Eugene et al., 2002), Marshall-Olkin Kumaraswamy exponential (MOKw-E) (Handique *et al*., 2017), Kumaraswamy Marshall-Olkin exponential (KwMO-E) (Alizadeh *et al*., 2015) and Kumaraswamy Poisson exponential (KwP-E) (Chakraborty *et al*., 2022) distribution.
Also considered some well-known model selection criteria namely the AIC, BIC, CAIC and HQIC and the Kolmogorov-Smirnov (K-S) statistics, Anderson-Darling (A) and Cramer von-mises (W) for goodness of fit to compare the fitted models also provided the asymptotic standard errors and confidence intervals of the mles of the parameters for each competing model. Visual comparison fitted density and the fitted cdf are presented in Figure 3. These plots reveal that the proposed distributions provide a good fit to this data. Here considered one failure time data set of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960).

- **TTT, Box plot and Descriptive Statistics for the failure time data:**

The total time on test (TTT) plot (see Aarset, 1987) is a technique to extract the information about the shape of the hazard function. A straight diagonal line indicates constant hazard for the data set, where as a convex (concave) shape implies decreasing (increasing) hazard. The TTT plots for the data sets Fig. 2 indicate that the data set have increasing hazard rate also provide the box plot of the data to summerise the minimum, first quartile, median, third quartile, and maximum where a box is shown from the first quartile to the third quartile with a vertical line going through the box at the median.

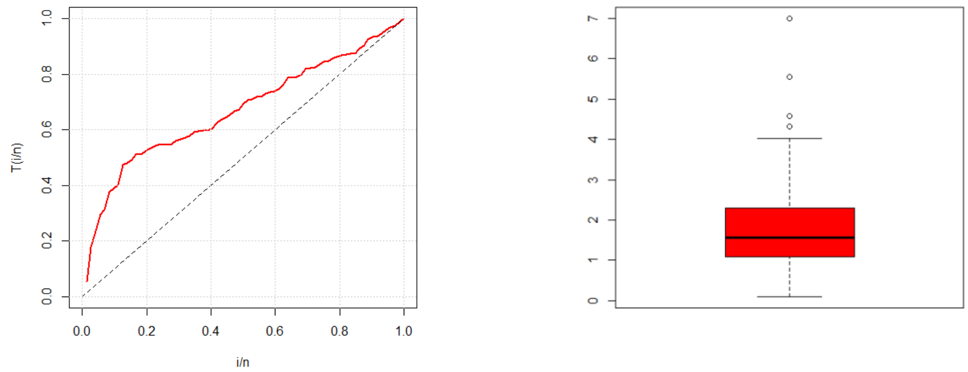


Figure: 2 TTT and Box plot for the failure time data set

Table 1: Descriptive Statistics for the failure time data set

Data Set	n	Min.	Mean	Median	s.d.	Skewness	Kurtosis	1[st] Qu.	3[rd] Qu.	Max.
I	72	0.100	1.851	1.560	1.200	1.788	4.157	1.080	2.303	7.000

Table 2: MLEs, standard errors, confidence intervals (in parentheses) values for the failure time data set

Models	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\beta}$
Exp	---	---	---	---	0.540
(β)					(0.063)
					(0.42, 0.66)
ME	---	---	---	---	0.925
(β)					(0.077)
					(0.62, 1.08)
MO-E	---	8.778	---	---	1.379
(α, β)		(3.555)			(0.193)
		(1.81,15.74)			(1.00,1.75)
GMO-E	0.179	47.635	---	---	4.465
(λ, α, β)	(0.070)	(44.901)			(1.327)
	(0.04, 0.32)	(0, 135.64)			(1.86. 7.07)
Kw-E	---	---	3.304	1.100	1.037
(a, b, β)			(1.106)	(0.764)	(0.614)
			(1.13,5.47)	(0, 2.59)	(0, 2.24)

B-E	---	---	0.807	3.461	1.331
(a,b,β)			(0.696)	(1.003)	(0.855)
			(0, 2.17)	(1.49,5.42)	(0, 3.01)
MOKw-E	---	0.008	2.716	1.986	0.099
(α,a,b,β)		(0.002)	(1.316)	(0.784)	(0.048)
		(0.004,0.01)	(0.14, 5.29)	(0.449, 3.52)	(0, 0.19)
KwMO-E	---	0.373	3.478	3.306	0.299
(α,a,b,β)		(0.136)	(0.861)	(0.779)	(1.112)
		(0.11, 0.64)	(1.79, 5.17)	(1.78, 4.83)	(0, 2.48)
KwP-E	4.001	---	3.265	2.658	0.177
(a,b,β,λ)	(5.670)		(0.991)	(1.984)	(0.226)
	(0, 15.11)		(1.32, 5.21)	(0, 6.55)	(0, 0.62)
MOKw-EP	1.155	1.465	2.038	1.687	0.114
$(\alpha,a,b,\beta,\lambda)$	(0.241)	(0.361)	(0.162)	(0.112)	(0.033)
	(0.68, 1.62)	(0.75, 2.17)	(1.72, 2.35)	(1.46, 1.91)	(0.04, 0.17)

Table 3: Log-likelihood, AIC, BIC, CAIC, HQIC, A, W and KS (*p*-value) values for failure time data set

Models	AIC	BIC	CAIC	HQIC	A	W	KS (*p*-value)
Exp (β)	234.63	236.91	234.68	235.54	6.53	1.25	0.27 (0.06)
ME (β)	210.40	212.68	210.45	211.30	1.52	0.25	0.14 (0.13)
MO-E (α,β)	210.36	214.92	210.53	212.16	1.18	0.17	0.10 (0.43)
GMO-E (λ,α,β)	210.54	217.38	210.89	213.24	1.02	0.16	0.09 (0.51)
Kw-E (a,b,β)	209.42	216.24	209.77	212.12	0.74	0.11	0.08 (0.50)
B-E (a,b,β)	207.38	214.22	207.73	210.08	0.98	0.15	0.11

							(0.34)
MOKw-E (α, a, b, β)	209.44	218.56	210.04	213.04	0.79	0.12	0.10
							(0.44)
KwMO-E (α, a, b, β)	207.82	216.94	208.42	211.42	0.61	0.11	0.08
							(0.73)
KwP-E (a, b, β, λ)	206.63	215.74	207.23	210.26	0.48	0.07	0.09
							(0.79)
MOKw-EP $(\alpha, a, b, \beta, \lambda)$	202.42	213.77	203.32	206.92	0.45	0.04	0.07
							(0.83)

The MLE's of the parameters with corresponding standard errors in the parentheses for all the fitted models along are given in Table 2 for the data set. While the various model selection criteria namely the AIC, BIC, CAIC, HQIC, A, W and KS statistic with a *p*-value for the fitted models of the data sets are presented in Table 3. From these findings based on the lowest values different criteria the MOKw-EP is found to be a better model than the models Exp, ME, MO-E, GMO-E, Kw-E, B-E, MOKw-E, KwMO-E and Kw-PE for the data set. A visual comparison of the closeness of the fitted density with the observed histogram and fitted cdf with the observed ogive for the data sets I are presented in the Figure 3 also indicate that the proposed distributions provide comparatively closer fit to this data set.

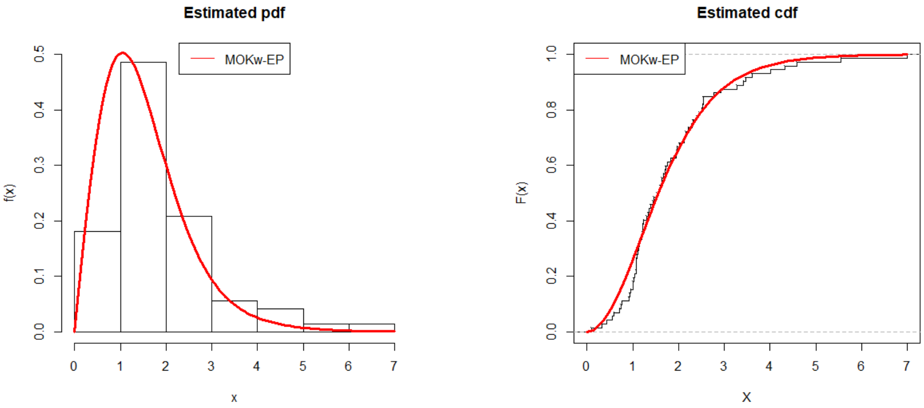


Figure 3: Plots for the epdf and ecdf of the MOKw-EP model for failure time data set

6. CONCLUSION

A new extension of Marshall-Olkin Kumaraswamy generalized family of distributions introduced which includes some well-known distribution and some of its important mathematical and statistical properties are studied. The maximum likelihood method for estimating the parameters are also discussed. Comparative data modelling application of the proposed model with some of its sub-models and other recently introduced models is carried out considering one failure time data set reveal its superiority.

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