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ROBUSTNESS OF BLOCK DESIGNS AGAINST THE INTERCHANGE OF TWO PAIRS OF TREATMENTS

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Conditions for robustness of binary variance balanced block designs agaist interchange of two pairs of treatments in two blocks of the design using connectedness and relative efficiency criterion has been derived.

1. Introduction :

During the layout of a planned experiment, each of the pair of experimental units belonging to different block may receive, by accident, treatment originally designed for the other i.e. interchanges may occur. Such discrepancy was first shown by Pearce (1948). Gomez and Gomez (1976) have listed this type of error in the application of treatments. Pearce (1983) has discussed the analysis of an experiment in randomized complete block (RCB) design with interchange of treatments occurring in two of its replications. The design may loose its optimal perorations under such disturbance. Several studies have been undertaken on the robustness of block designs against (1) missing observations (2) presence of outliers (3) model inadequacy (4) presence of systematic trends among observation with in a block. Batra et. al. (1997) investigated the robustness of block designs against interchange of one pair of treatments using (1) connectedness property Ghosh (1979) of the resulting design (2) Relative efficiency John (1976) of the resulting design with respect to the original design, and posed a problem in his paper regarding study of robustness aspects of VBD's when there is interchange of more than one pair of treatments. In the present paper, authors attempted this problem by taking two pairs interchanged accidentally between two different block VB proper Blocks designs.

For the present investigation, under general proper block designs we have obtained the informatiom matrix C* of the resulting design (d*) in terms of the original design parameters in section 2. The eigen values of C* when design d is binary variance balance d block design have been obtained in section 3. The robustness of RCB designs have been investigated in section.

2. Interchange of treatments in a proper block design

Consider a binary block design d (v, b, r, k) in which tratments are allotted to experimental units arranged in b blocks with block size vector $\mathbf{K}^1 = (k_1, k_2, ... k_b)$ and replication vector $\mathbf{r}^1 = (r_1, r_2, ... r_v)$ Let $N = (n_{ij})$ be a v x b incidence matrix.

The v x v information matrix.

$$C = R - N K^{-1} N^1 \quad ... \quad (2.1)$$

Where $R = \text{diag}(r_1, r_2, ... r_v)$ and $\mathbf{K} = \text{diag}(k, k, ...k_h)$. For a connected block design Rank (C) = v − 1 we shall denote the v tretments in the disign d as 1, 2,..... v and b blocks as 1,2,......b. Let treatment i, in block j_1, and treatment I_3 in block j_1 be interchanged with treatment i_2 and treatment i_4 respectively in block j_2. There will be total of 16 cases depending on the presence and absence of particular treatment in the two blocks.

We shall denote the resulting design by d* four treatments 1,2,3 and 4, in the interchange as affected treatments and other (v-4) treatment as unaffected treatments, similarly the blocks involved in the interchange of the treatments are termed as the affected block and other (b-2) blocks as unaffected blocks.

In order to investigate the possible forms of **N***, the incidence matrix of d*, we rewrite the incidence matrix **N** with elements, 0, 1 of designd as

$$\mathbf{N} = [\, \mathbf{n}_1, \mathbf{n}_2, \mathbf{N}_1] \quad ... \quad (2.2)$$

$$\text{where } n_1 = \begin{pmatrix} 1 \\ x_1 \\ 1 \\ y_1 \\ u_1 \end{pmatrix} \qquad \mathbf{n}_2 = \begin{pmatrix} x_2 \\ 1 \\ y_2 \\ 1 \\ u_2 \end{pmatrix} \quad ... \quad (2.3)$$

x_1 = 1 if trearment 2 is present in block 1
= 0 otherwise.

x_2 = 1 if treatment 1 is present in block 2
= 0 otherwise.

y_1 = 1 if treatment 4 is present in block 1
= 0 otherwise.

y_2 = 1 if treatment 3 is present in block 2
= 0 otherwise.

u_i = (v-4) x 1 incidence sector of (v-4) unaffected treatments in block t (t = 1.2). Thus depending upon the incidence of interchanged treatments in the two affectec blocks, 16 cases will be generated. Four cases are presented for the illustration.

$$n_1 = \underset{\text{Case I}}{\begin{pmatrix}1\\0\\1\\0\\u_1\end{pmatrix}} \quad \underset{\text{Case II}}{\begin{pmatrix}1\\0\\1\\1\\u_1\end{pmatrix}} \quad \underset{\text{Case III}}{\begin{pmatrix}1\\0\\1\\1\\u_1\end{pmatrix}} \quad \underset{\text{Case IV}}{\begin{pmatrix}1\\0\\1\\1\\u_1\end{pmatrix}} \quad \cdots \tag{2.4}$$

$$n_2 = \underset{\text{Case I}}{\begin{pmatrix}0\\1\\0\\1\\u_2\end{pmatrix}} \quad \underset{\text{Case II}}{\begin{pmatrix}0\\1\\1\\1\\u_2\end{pmatrix}} \quad \underset{\text{Case III}}{\begin{pmatrix}0\\1\\0\\1\\u_2\end{pmatrix}} \quad \underset{\text{Case IV}}{\begin{pmatrix}0\\1\\1\\1\\u_2\end{pmatrix}}$$

Remark

In a given block design it may not always possible to have all the choices of the affected blocks i. e for a BIBD d (v, b, r, k, λ) with λ=1 it is not possible to have a pair of block for case II presented in 2.4 In the remaining cases also some cases can be obtained from the other simly by the renumbering of treatmets and blocks.

Let c be the total number of common tretments in the two affected blocks of which r is the number of unaffected treatments, thus $n_1^1 n_2 = c$ and $u_1^1 u_2 = r$ these v treatments can be grouped as given in **Table 1**.

Table 1

Group	Nature of grouping	No. of treatments
1.	Affected treatment	4
2.	Common unaffected treatment in the affected block.	p
3.	Unaffected treatments in block 1 and not in block 2.	$k-c-1 + x_2 + y_2$ (= v_1 say)
4.	Unaffected treatments in block 2 and not in block 1.	$k-c-1 + x_1 + y_1$ (= v_1 say)
5.	Treatments neither occurring in block 1 nor in block 2	$v - k - k + c$(= v_0 say)

Here, $v_0 + v_1 + v_2 + r = v - 4$ without loss of generality after suitably renumbering if necessary, the vectors n_1, n_2 defined in 2.3 can be written as.

$$n_1 = \left(1 \quad x_1 \quad y_1 \quad 1 \quad 1'_\rho \quad 1'_{v_1} \quad 0'_{v_2} \quad 0'_{v_0}\right)$$

$$n_2 = \left(x_2 \quad 1 \quad y_2 \quad 1 \quad 1'_\rho \quad 0'_{v_1} \quad 1'_{v_2} \quad 0'_{v_0}\right) \qquad (2.5)$$

The rows of **N** and **N*** defined in Table 1 are also rearranged according to the arrangement of treatments in vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ in **2.5** we have the following results

Lemma :-

i) $\mathbf{n_1}^1\mathbf{n_1} = \mathbf{n_2}^1\mathbf{n_2} = k$

ii) $\mathbf{u_1}^1\mathbf{u_1} = k - 2 - (x_1 + y_1)$

$\mathbf{u_2}^1\mathbf{u_2} = k - 2 - (x_2 + y_2)$

iii) $\mathbf{n_1}^1\mathbf{n_2} = \mathbf{u_1}^1\mathbf{u_2} + x_1 + x_2 + y_1 + y_2$ (2.6)

$$c = \rho + x_1 + x_2 + y_1 + y_2$$

iv) $\mathbf{NK}^{-1}\mathbf{N}^1 = (\mathbf{n_1 n_1}^1 + \mathbf{n_2 n_2}^1)\mathbf{K}^{-1} + \mathbf{N_1 K}^{-1}\mathbf{N_1}^1$

Where $\mathbf{K_1}$ = diag (k, k, ..., k)

Thus the incidence matrix **N*** of the resulting design d* can be written as.

$$\mathbf{N^*} = \left(\mathbf{n_1} - \mathbf{e_1} + \mathbf{e_2} \quad \mathbf{n_2} - \mathbf{e_2} + \mathbf{e_1} \quad \mathbf{N_1}\right)$$

Where $\mathbf{e_1} = \left(1, 0, 1, 0, \mathbf{0}^1\ \mathbf{v\text{-}4}\right)^1$, $\mathbf{e_2} = \left(0, 1, 0, 1, \mathbf{0}^1\ \mathbf{v\text{-}2}\right)^1$ (2.7)

The parameter of d* are same as that of d with incidence matrix as given in Table 2.7. Thus, the information matrix d* is

$$C^* = R - N^*K^{-1}N^{*1} \qquad (2.8)$$

Using lemma it can easily be seen that

$$\mathbf{C^* = C - A}$$

Where : ... (2.9)

A =

$$\begin{pmatrix} 2x_2/k & -(x_1+x_2)/k & (x_2+y_2)/k & -(y_1+x_2)/k & -u_1^1/k+u_2^1/k \\ -(x_1+x_2)/k & 2x_1/k & -(x_1+y_2)/k & (x_1+y_1)/k & u_1^1/k-u_2^1/k \\ (x_2+y_2)/k & -(x_1+y_2)/k & 2y_2/k & -(y_1+y_2)/k & -u_1^1/k+u_2^1/k \\ -(y_1+x_2)/k & (x_1+y_1)/k & -(y_1+y_2)/k & 2y_1/k & u_1^1/k-u_2^1/k \\ -u_1^1/k+u_2^1/k & u_1^1/k-u_2^1/k & -u_1^1/k+u_2^1/k & u_1^1/k-u_1^1/k & 0 \end{pmatrix}$$

For further investigation, we shall assume design d to be a binary variance balanced design.

3. Eigen values of C*

For a connected binary variance balanced block design d (v, b, r, k) the information matrix is $C = \mu\left(1-(1/v)\, 1_v\, 1_v^1\right)$ where μ is unique non-zero eigenvalue of C. Hence the information matrix of d* using 2.1 is

$$C^* = \mu\left(1-(1/v)\, 1_v\, 1_v^1\right) - A \quad \ldots \quad (3.1)$$

The eigen values of A for the RBD case are.

(i) $\theta_0 = 0$ with multiplicity (v-2) ... (3.2)

(ii) $\theta_1 = 8/K$

Therefore, efficiency (E) of d* relative to d, is the ratio of harmonic mean of non-zero eigen values of C* to that of C and given by.

$$E = (v\text{-}1)\ \theta_2^* \,/\, \left((v-1)\,\theta_2^* + \theta_1^*\right) \quad \ldots \quad (3.3)$$

Particular case.

The eigen values of information matrix C* are.

(i) 0 with multiplicity one.

(ii) $\theta_1^* = \mu$ with multiplicity (v - 2)

iii) $\theta_2^* = (\mu - 8/k)$ with multiplicity one.

For a RCB design with v treatments and r replications, the unique non-zero eigen value of C is r with multiplicity v-1. Then the non-zero eigen value of C* of d* using 3.1 are.

(i) 0 with multiplicity one.

(ii) r with multiplicity (v-2)

(iii) (r-8 / v) with multiplicity one.

Therefore d* is connected if $r \neq 8 / v$.

Randomized complete block designs except r = 8 / v are robust against the interchange of two pair of treatments in two blocks according to connectedness cirterion.

It is observed that for design r > 4 and v > 8 the efficiency is greater than 95%. The minimum number of replications required for the efficiency to be more than 95% in case of RCB designs with 4<v<8 are as given below.

v	:	5	6	7	8
min r	:	10	7	5	4

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